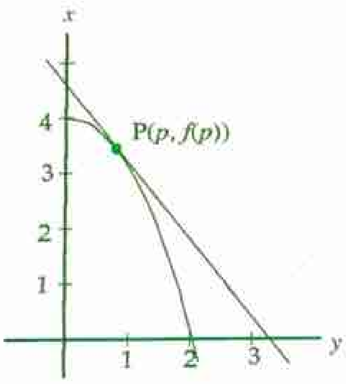
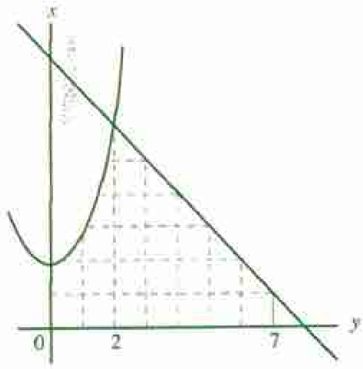
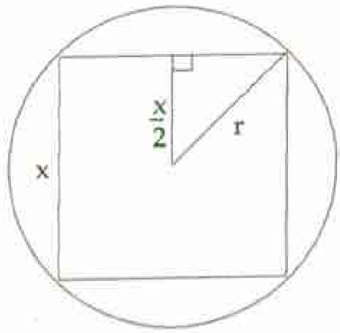
Complete Solutions Manual to Accompany

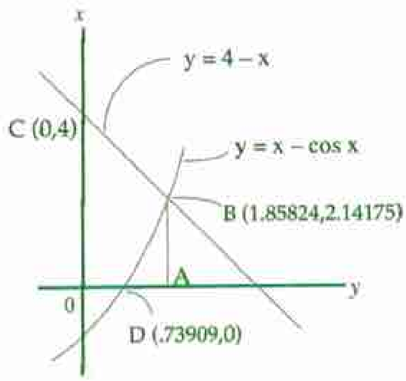
Preparing for the

AP Calculus (BC) Examination









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Exam I
Section I
Part A — No Calculators

1. D p. 1

$$\int_0^5 \frac{dx}{\sqrt{3x+1}} = \frac{1}{3}\int_0^5 (3x+1)^{-1/2}(3\ dx) = \frac{1}{3}\frac{(3x+1)^{1/2}}{1/2}\Bigg|_0^5 = \frac{2}{3}[4-1] = 2$$

2. C p. 1

Is f defined at $x = 0$ and is $\lim_{x\to 0} f(x) = f(0)$?

I. $f(0) = |0| = 0$ and $\lim_{x\to 0} |x| = f(0)$ — Continuous

II. $e^0 = 1$ and $\lim_{x\to 0} e^x = 1$ — Continuous

III. $\ln(e^0 - 1) = \ln(0)$ which is undefined — Not continuous

3. B p. 2

$y = x^2 + \frac{c}{x} = x^2 + cx^{-1}$

$y' = 2x - cx^{-2} \Rightarrow y'(-1) = -2 - c.$ When $-2 - c = 0$, then $c = -2$.

$y'' = 2 + 2cx^{-3} \Rightarrow y''(-1) = 2 + 4 = 6$ and $(-1, 3)$ is a relative minimum.

4. C p. 2

The curves are clearly periodic so the only choices are sine or cosine.
At $x = 0$, cosine has a horizontal tangent, so the answer is C.

5. D p. 2

First we have to find the intersection points of the two curves.

$3x^2 = 6x \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

Hence the curves intersect at (0,0) and (2,12).

Cross sections of the solid taken perpendicular to the x-axis are "washers." The volume of the solid is given by:

$$V = \pi\int_0^2 [(6x)^2 - (3x^2)^2]\ dx = \pi\int_0^2 (36x^2 - 9x^4)\ dx$$

6. E p. 3

$f'(x) = e^{\sin x}$

I. $f''(x) = e^{\sin x} \cos x \; f''(0) = 1 \cdot 1 = 1$ TRUE

II. Slope of $y = x + 1$ is 1, and the line goes through (0, 1) TRUE

III. $h(x) = f(x^3 - 1)$

$h'(x) = f'(x^3 - 1) \cdot (3x^2) = 3x^2 e^{\sin(x^3 - 1)}$

Since $h'(x) \geq 0$ for all x, then the graph of $h(x)$ is increasing. TRUE

---

7. C p. 3

$$\int_1^\infty \frac{3x^2}{(1 + x^3)^2}\,dx = \lim_{a\to\infty} \int_1^a (1 + x^3)^{-2} (3x^2\,dx) = \lim_{a\to\infty} \left[-\frac{1}{1 + x^3}\right]_1^a$$

$$= \lim_{a\to\infty} \left[-\frac{1}{1 + a^3} + \frac{1}{2}\right] = \frac{1}{2}$$

---

8. A p. 3

Let R denote the ratio of successive terms of the series: $\frac{a_{n+1}}{a_n}$.

Then $|R| = \left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^{n+1}(x-2)^n}\right| = \left|\frac{(-1)(x-2)}{3} \frac{n}{n+1}\right|$.

Hence $\lim_{n\to\infty} |R| = \left|\frac{x-2}{3}\right|$.

For convergence, we must have $\left|\frac{x-2}{3}\right| < 1$, so that $|x - 2| < 3$.

Hence the radius of convergence is $r = 3$.

---

9. A p. 4

The slope of PQ $= \frac{1}{e - 1}$.

The slope of the tangent to the graph of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$.

We need the slope of the tangent equal to the slope of PQ.

Hence $\frac{1}{x} = \frac{1}{e - 1}$. Therefore $x = e - 1$

---

10. D p. 4

Using implicit differentiation on $4x^2 + 2xy + 3y = 9$

gives $8x + 2y + 2xy' + 3y' = 0$.

At $(2, -1)$: $16 - 2 + 4y' + 3y' = 0 \Rightarrow y' = -2$.

---

11. C p. 4

Solution I. $\frac{dy}{dx} = \sqrt{x} \Rightarrow y = \frac{2}{3}x^{3/2} + C$

Then the average rate of change of y on the interval [0,4] is

$$\frac{y(4) - y(0)}{4 - 0} = \frac{(\frac{2}{3} \cdot 8 + C) - C}{4} = \frac{4}{3}$$

Solution II. $y_{ave} = \frac{1}{b - a}\int_a^b f(x)\,dx$

$$\text{ave}\frac{dy}{dx} = \frac{1}{4 - 0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{1}{4}\left.\frac{x^{3/2}}{3/2}\right|_0^4 = \frac{1}{6}[8 - 0] = \frac{4}{3}$$

---

12. B p. 5

$x(t) = \cos(2t) + \sec t$

$v(t) = x'(t) = -2\sin(2t) + (\sec t)(\tan t)$

$a(t) = x''(t) = -4\cos(2t) + \sec^3 t + (\sec t)(\tan^2 t)$

When $t = \pi$, the acceleration is $a(\pi) = -4(1) + (-1)^3 + (-1)(0) = -5$.

---

13. E p. 5

Solution I. $\int_0^p \sin x\,dx = \int_p^\pi \sin x\,dx + 1$

$-\cos x\Big|_0^p = -\cos x\Big|_p^\pi + 1$

$-\cos p + \cos 0 = -\cos \pi + \cos p + 1 \Rightarrow \cos p = -\frac{1}{2} \Rightarrow p = \frac{2\pi}{3}$

Solution II. $\int_0^\pi \sin x\,dx = 2.$

If the area under $y = \sin x$ to the left of $x = p$ is to be 1 more than the area to the right of $x = p$, then the area to the left is 1.5 while the area to the right is 0.5.

Hence we need to determine p such that $\int_0^p \sin x\,dx = 1.5$.

$\int_0^p \sin x\,dx = 1.5 \Rightarrow -\cos x\Big|_0^p = 1.5$

$\Rightarrow -\cos p + \cos 0 = 1.5$

$\Rightarrow -\cos p + 1 = 1.5$

$\Rightarrow \cos p = -.5$

$\Rightarrow p = \frac{2\pi}{3}$

---

14. B p. 5

$f(x) = \ln x,\ f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{2}{x^3}$

$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2;\ a_0 = 0,\ a_1 = 1,\ a_2 = -\frac{1}{2},\ a_3 = \frac{1}{3};$

$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

---

15. E p. 6

$h(x) = [f(x)]^2 + f(x)g(x),\ f'(x) = g(x)$ and $g'(x) = -f(x)$, then

$h'(x) = 2f(x)f'(x) + f(x)g'(x) + f'(x)g(x)$

$h'(x) = 2f(x)g(x) - f(x)f(x) + g(x)g(x)$

---

16. D p. 6

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \text{Arcsin}\frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{\frac{1}{2}}{\sqrt{1 - \frac{x^2}{4}}} = \frac{1}{\sqrt{4 - x^2}}$$

$$L = \int_a^b \sqrt{1 + \frac{1}{4 - x^2}}\, dx$$

---

17. E p. 7

$\frac{dy}{dx} > 0$ for all x implies that the graph of y is increasing.

$\frac{d^2y}{dx^2} < 0$ for all x implies that the graph of y is concave down.

(E) is the only choice satisfying both conditions.

---

18. D p. 7

Using the method of partial fractions, $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x+1) + Bx$

Then $x = 0 \Rightarrow A = 1$
and $x = -1 \Rightarrow B = -1$

$$\int \frac{1}{x^2 + x}\, dx = \int \left(\frac{1}{x} + \frac{-1}{x+1}\right) dx$$

$$= \ln|x| - \ln|x+1| + C$$

$$= \ln\left|\frac{x}{x+1}\right| + C$$

---

19. A p. 8

$f(x) = x^2 e^{-2x}$

$f'(x) = 2xe^{-2x} - 2x^2 e^{-2x} = 2xe^{-2x}(1 - x) = \dfrac{2x(1-x)}{e^{2x}}$.

Since $f'(x) > 0$ for $0 < x < 1$, the graph of f is increasing there.

---

20. D p. 8

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$$

Since this has both odd and even powers of x, it is neither $\sin x$ nor $\cos x$.

Since $f'(x) = -1 + x - \dfrac{x^2}{2} + \ldots = -f(x)$, this suggests e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \text{ and}$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots = f(x)$$

---

21. C p. 8

$$\int_{-\pi/4}^{e-1} f(x)\,dx = \int_{-\pi/4}^{0} f(x)\,dx + \int_{0}^{e-1} f(x)\,dx = \int_{-\pi/4}^{0} \sec^2 x\,dx + \int_{0}^{e-1} \frac{1}{x+1}\,dx$$

$$= \tan x\Big|_{-\pi/4}^{0} + \ln|x+1|\Big|_{0}^{e-1}$$

$$= (0 - (-1)) + (\ln e - \ln 1) = 2$$

---

22. C p. 9

Let $y = \left(1 + \frac{x}{2}\right)^{\cot x}$. As $x \to \infty$, this takes on the indeterminate form 0^{∞}.

Taking the logarithm of both sides gives $\ln y = (\cot x)\ln(1 + \frac{x}{2})$.

$$\lim_{x\to 0}[\ln y] = \lim_{x\to 0}[(\cot x)\ln(1 + \tfrac{x}{2})] = \lim_{x\to 0}\left[\frac{\ln(1+\frac{x}{2})}{\tan x}\right] = \frac{0}{0}$$

Using L'Hôpital's Rule: $$\lim_{x\to 0}[\ln y] = \lim_{x\to 0}\left[\frac{\frac{1}{1+\frac{x}{2}}\cdot\frac{1}{2}}{\sec^2 x}\right] = \frac{\frac{1}{2}}{1} = \frac{1}{2}.$$

If f is continuous, then $\lim[f(g(x))] = f[\lim g(x)]$. Since the ln function is continuous, we have $\ln[\lim y] = \frac{1}{2}$, so that $\lim\limits_{x\to 0} y = e^{1/2}$.

---

23. D p. 9

In polar coordinates, Area $= \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$

$r = \sqrt{1+\cos\theta} \quad \Rightarrow \quad r^2 = 1 + \cos.$

Using the symmetry of cosine, $A = \frac{1}{2}\cdot 2\int_0^{\pi}(1 + \cos\theta)\, d\theta = \int_0^{\pi}(1 + \cos\theta)\, d\theta$

---

24. C p. 10

Using the definition of the derivative, $\lim_{h\to 0}\left[\frac{(3+h)^5 - 3^5}{9h}\right] = \frac{1}{9} f'(3)$

where $f(x) = x^5$ and $f'(x) = 5x^4$.

Thus $\frac{1}{9} f'(3) = \frac{1}{9}\cdot 5\cdot 3^4 = 45$

---

25. D p. 10

Since the three subintervals indicated by the table have uneven lengths, it is necessary to calculate the area of the individual trapezoids. Using the function values from the table:
$f(0) = 3, f(4) = k, f(7) = 9$ and $f(9) = 11$ we have

$\frac{1}{2}(4)[3+k] + \frac{1}{2}(3)[k+9] + \frac{1}{2}(2)[9+11] = 57$

$(6+2k) + (\frac{3}{2}k + \frac{27}{2}) + (20) = 57$

$12 + 4k + 3k + 27 + 40 = 114$

$7k = 35$

$k = 5$

---

26. E p. 10

Rewrite the integrand: $\int x\sqrt{1-x^2}\, dx = \int (1-x^2)^{1/2}\cdot x\, dx$.

The factor $(1-x^2)^{1/2}$ is a composite function $f'(g(x))$ an outer function $f'(x) = x^{1/2}$ and an inner function $g(x) = 1 - x^2$. The derivative of the inner function, $g'(x) = -2x$, appears as a factor in the integrand except for the constant factor -2. This can be remedied.

$$\int (1-x^2)^{1/2}\cdot x\, dx = -\frac{1}{2}\int (1-x^2)^{1/2}\cdot(-2x)\, dx$$
$$= -\frac{1}{2}\int f'(g(x))\cdot g'(x)\, dx$$
$$= -\frac{1}{2} f(g(x)) + C, \text{ where } f \text{ is an antiderivative of } f'.$$

Since an antiderivative of $f'(x) = x^{1/2}$ is $f(x) = \frac{2}{3}x^{3/2}$, this gives

$$\int (1-x^2)^{1/2}\cdot x\, dx = -\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$$

---

27. C p. 11

From the table, $\Delta x = 0.5$.

Then $f(1.5) \approx f(1) + f'(1)\,\Delta x = 2.0 + (0.4)(0.5) = 2.2$.

The second step gives $f(2.0) \approx f(1.5) + f'(1.5)\,\Delta x = 2.2 + (0.6)(0.5) = 2.5$.

---

28. B p. 11

$$\frac{dy}{dx} = y^2$$

$$\frac{1}{y^2}\,dy = dx$$

$$-\frac{1}{y} = x + C \qquad \text{At } (-1, 1) \quad -1 = -1 + C \text{ and } C = 0$$

$$-\frac{1}{y} = x + 0 \Rightarrow y = -\frac{1}{x}$$

The solution to a differential equation must be a function that contains the initial condition and has a continuous domain.

The largest domain that contains (-1, 1) is $x < 0$.

Exam I
Section I
Part B — Calculators Permitted

1. C p. 12

$y = x^2 - e^{-x} \Rightarrow y' = 2x + e^{-x} \Rightarrow y'' = 2 - e^{-x}$

$y'' = 0 \Rightarrow 2 - e^{-x} = 0 \Rightarrow e^x = \frac{1}{2}$

Then $x = -\ln 2$

$y'(-\ln 2) = 2(-\ln 2) + e^{\ln 2} = -2\ln 2 + 2 = 2 - \ln 4$

---

2. C p. 12

The slope of the parabola $y = 2x^2 + x + k$ at any point is given by $y' = 4x + 1$.
The slope of the line is $m = -3$. Since the line is tangent to the parabola, these slopes must be equal.
$4x + 1 = -3 \Rightarrow x = -1$. From the equation of the line we learn then that $y = 4$.
From the parabola $4 = 2 - 1 + k \Rightarrow k = 3$.

---

3. D p. 13

I. $f(1) = \int_0^1 \frac{1}{1+t^4}\,dt \approx 0.867$ False

II. By the Fundamental Theorem, $f'(x) = \frac{1}{1+x^4}$.

Then $f''(x) = \frac{-4x^3}{(1+x^4)^2}$; $f''(3) = -0.016$, so that $f''(3) < 0$ and f is concave down.

True

III. $f(-x) = \int_0^{-x} \frac{1}{1+t^4}\,dt = \int_x^0 \frac{1}{1+t^4}\,dt$ because $\frac{1}{1+t^4}$ is even.

Hence $f(-x) = -\int_0^x \frac{1}{1+t^4}\,dt = -f(x)$. Thus $f(-x) + f(x) = 0$. True

---

4. B p. 13

Using the definition of the derivative, $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \sec^2(3x)$.

Thus $f(x) = \frac{1}{3}\tan(3x) + C$. Since $f(0) = 1$, $1 = \frac{1}{3}\tan(0) + C \Rightarrow C = 1$.

$\int_0^{.5} \left(\frac{1}{3}\tan(3x) + 1\right) dx \approx 0.794$

---

5. D p. 13

I. $h(2) = f(g(2)) = f(4) = 3$ False

II. $h'(x) = f'(g(x)) \cdot g'(x)$

These derivatives are approximated by estimating the slopes of lines that are tangent to the graphs of f and g. Thus $h'(4) = f'(g(4)) \cdot g'(4) = f'(1) \cdot g'(4) \approx (-1)(-1) = 1$ True

III. $h'(1) = f'(g(1)) \cdot g'(1) = f'(3) \cdot 0 = 0$ True

---

6. E p. 14

The graph of v shows that the particle moves both left and right.

$$\text{Distance} = \int_0^4 |v(t)|\,dt = \int_0^4 \left|\cos(t + \sqrt{t})\right|\,dt \approx 2.416$$

---

7. B p. 14

Use a graphing calculator to find the intersection points of the two functions $y = \arctan x$ and $y = 4 - x^2$.

They are: $x_1 = -2.271$ and $x_2 = 1.719$.

$$\text{Then } A = \int_{-2.271}^{1.719} (4 - x^2 - \arctan x)\,dx \approx 10.972$$

---

8. D p. 14

I. The given series converges by the Alternating Series Test.

However, $\sum_{n=1}^{\infty} \frac{1}{2n+1} \ge \sum_{n=1}^{\infty} \frac{1}{3n}$ and $\sum_{n=1}^{\infty} \frac{1}{3n}$ is divergent.

Hence the given series is conditionally convergent.

II. The given series is **not** alternating, in spite of the $(-1)^n$. The factor $\cos n$ in each fraction is itself sometimes positive and sometimes negative.

Since $|\cos n| \le 1$ for all n, $\frac{|\cos n|}{3^n} \le \frac{1}{3^n}$ for all $n \ge 1$.

Hence $\sum_{n=1}^{\infty} \frac{|\cos n|}{3^n} \le \sum_{n=1}^{\infty} \frac{1}{3^n}$ (a convergent geometric series).

The given series converges absolutely.

III. The given series converges by the Alternating Series Test.

We note that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{.5}}$ (a divergent p-series).

Hence the given series is conditionally convergent.

---

9. A p. 15

$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$

$y = \ln(t^2 + 1) \Rightarrow \frac{dy}{dt} = \frac{1}{t^2 + 1} \cdot 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{t^2 + 1}$$

$$\frac{d^2y}{dx^2} = \frac{D_t\left(\frac{dy}{dx}\right)}{D_t(x)} = \frac{\frac{-2t}{(t^2+1)^2}}{2t} = \frac{-1}{(t^2+1)^2}$$

Then $\left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{-1}{(t^2+1)^2}\right|_{t=1} = -\frac{1}{4}$

---

10. B p. 15

The relationship between the supply x and price p is given by the equation $px - 20p - 6x + 40 = 0$.

Differentiating with respect to time t, we have $p\frac{dx}{dt} + x\frac{dp}{dt} - 20\frac{dp}{dt} - 6\frac{dx}{dt} + 0 = 0$.

From the original equation, we determine p when $x = 100$.

$100p - 20p - 6 \cdot 100 + 40 = 0 \Rightarrow 80p = 560 \Rightarrow p = 7$

We are given that when $x = 100$, the supply is **decreasing** at a rate of 8 crates per day, so that $\frac{dx}{dt} = -8$.

Substituting $x = 100$, $p = 7$, $\frac{dx}{dt} = -8$ into the equation with derivatives yields

$7(-8) + 100\frac{dp}{dt} - 20\frac{dp}{dt} - 6(-8) = 0$.

Hence $80\frac{dp}{dt} = 8$, so $\frac{dp}{dt} = .1$.

The price is increasing at \$0.10 per day.

---

11. E p. 15

The two curves intersect at $x = 0, \pm 2.331$. The first quadrant region is defined on the interval $[0, 2.331]$.

Using the washer method we obtain

$$V = \int_0^{2.331} \pi\left[(2\text{Arctan } x)^2 - x^2\right] dx \approx 7.151$$

---

12. B p. 16

$\mathbf{p}(t) = \langle e^t, e^{-t}\rangle$

$\mathbf{v}(t) = \langle e^t, -e^{-t}\rangle$

$\mathbf{v}(1) = \langle e, -\frac{1}{e}\rangle$

The speed at $t = 1$ is $|\mathbf{v}(1)| = \sqrt{e^2 + \frac{1}{e^2}} \approx 2.743$

---

13. A p. 16

$$y_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Using fnInt or graphing $L(x) = e^{-0.1x} + \frac{1}{x^2}$ for f(x) on a = 15 to b = 25, gives:

$$L_{ave} = \frac{1}{25-15}\int_{15}^{25} L(x)\,dx = 0.144 .$$

14. B p. 17

$$S = \sqrt{x^2+y^2} = \sqrt{x^2+e^{2x}}$$

$$\frac{dS}{dx} = \frac{2x + 2e^{2x})}{2\sqrt{x^2+e^{2x}}}$$

$$\frac{dS}{dx} = 0 \quad \Rightarrow \quad x = -e^{2x} .$$

We solve this equation with a graphing calculator to obtain $x \approx -0.426$.

Since $\frac{dS}{dx} < 0$ for $x < -0.426$ and $\frac{dS}{dx} > 0$ for $x > -0.426$, the minumum value of S is at x = - 0.426.

15. C p. 17

The Lagrange error is $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$.

$f(x) = e^{x/2}$	$f(0) = 1$	
$f'(x) = \frac{1}{2}e^{x/2}$	$f'(0) = \frac{1}{2}$	Since f is increasing, positive and continuous
$f''(x) = \frac{1}{4}e^{x/2}$	$f''(0) = \frac{1}{4}$	on [0,2], the maximum is at x = 2 is
$f'''(x) = \frac{1}{8}e^{x/2}$	$f'''(0) = \frac{1}{8}$	$f'''(x) = \frac{1}{8}e^{2/2} = f'''(x) = \frac{e}{8}$.

When n = 2, a = 0, and x = 2, $R_2(x) = \frac{\frac{e}{8}(2-0)^3}{3!} = 0.453$.

16. B p. 18

To antidifferentiate $\int f(x)\cos x\,dx$, use Integration by Parts.

Take $\left\{ \begin{array}{ll} u = f(x) & u' = f'(x) ; \\ v' = \cos x & v = \sin x . \end{array} \right\}$

Then $\int f(x)\cos x\,dx = f(x)\sin x - \int f'(x)\sin x\,dx$.

Since we are given that $\int f(x)\cos x\,dx = f(x)\sin x - \int 6x^2 \sin x\,dx$, this means that $f'(x) = 6x^2$. Thus $f(x) = 2x^3 + C$.

The only proposed answer of this form is $f(x) = 2x^3$.

17. B p. 18

I. If the function is decreasing, the left-hand Riemann sums will be greater than the right-hand sums. Thus a function that is concave up but decreasing (for instance, $f(x) = x^2$ for $-1 \le x \le 0$) will have left-hand Riemann sums greater than the right-hand sums. **False**

II. To obtain a definite integral equal to 0, the area below the x-axis must equal the area above the x-axis. This can only happen if the function has points on each side of the x-axis (or if it is the constant 0 function). In either case, continuity (with the Intermediate Value Theorem) forces the function to have a zero between a and b. **True**

III. If $f'(x) > 0$ on an interval, then f is increasing on that interval.

To have the graph of f concave up on an interval, it is $f''(x)$ that must be positive-valued. **False**

Exam I
Section II
Part A—Calculators Permitted

1. p. 20

(a) Find the zeros of $y = -16t^2 + .05t + \sqrt{168}$
When $y = 0$, $t \approx 0.902$ sec.

$2: \begin{cases} 1\text{: sets } y(t) = 0 \\ 1\text{: solves for t} \end{cases}$

(b) The vertical velocity of the top of the ladder is given by $\frac{dy}{dt} = -32t + 0.05$

Differentiating $169 = x^2 + y^2$ with respect to time gives $0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$.

When $x = 5$, then $y = 12$ and we use the calculator to find the value of t when $12 = -16t^2 + .05t + \sqrt{168}$. Thus $t \approx 0.247$ sec and $\frac{dy}{dt} \approx -7.84457$ ft/sec.

Thus, $0 = 2(5)\frac{dx}{dt} + 2(12)(-7.84457)$ and $\frac{dx}{dt} = 18.827$ ft/sec.

$7: \begin{cases} 1: \frac{dy}{dt} \\ 1: y \text{ when } x = 5 \\ 1\text{: expression relating quantities} \\ 1\text{: implicit differentiation } 2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} = 0 \\ 1: \frac{dy}{dt} \text{ when } y = 12 \\ 2\text{: answer (with units)} \end{cases}$

2. p. 21

(a) $g(x) = (x^2 + 1)\cdot f(x)$. The product rule gives $g'(x) = 2x\,f(x) + (x^2 + 1)\cdot f'(x)$.
Hence, $g'(0) = 2\cdot 0\cdot 3 + 1\cdot f'(0) = 0 + 1 = 1$.

$2: \begin{cases} 1: g'(x) \\ 1: g'(0) \end{cases}$

(b) g is increasing if $g'(x) > 0$. Since $g'(x) = 2x\,f(x) + (x^2 + 1)\cdot f'(x)$, we have $g'(1) = 2\,f(1) + (2)\cdot f'(1) = 2\,[f(1) + f'(1)]$. From the graph, $f'(x) > 0$ for $-1 < x < 4$. This also means that f is increasing on $-1 < x < 4$. Since $f(0) = 3$, then $f(1) > 3$ and $f(1) + f'(1) > 3$.
Hence $g'(1) \geq 3$ so that g is increasing.

$2: \begin{cases} 1\text{: answer} \\ 1\text{: justification} \end{cases}$

(c) Differentiating $g'(x) = [2x\,f(x)] + (x^2 + 1)\cdot f'(x)$
gives $g''(x) = [2\,f(x) + 2x\,f'(x)] + 2x\,f'(x) + (x^2 + 1)\cdot f''(x)$.
Then $g''(0) = [2\cdot 3 + 2\cdot 0\cdot 1] + 2\cdot 0\cdot 1 + 1\cdot f''(x)$
The slope of the tangent to $f'(x)$ at $(0, f(0))$ is about 1; hence $f''(0) = 1$.
This gives $g''(0) = 6 + 1 = 7$

$3: \begin{cases} 1: g''(x) \\ 1\text{: estimate of } f''(0) \\ 1\text{: answer} \end{cases}$

(d) The graph of g is concave up at $x = 1$ if $g''(1) > 0$.
Since $f(1) > 3$, $f'(1) > 0$ and $f''(1) > 0$,
then $g''(1) = 2f(1) + 4f'(1) + 2f''(1) > 0$.

$2: \begin{cases} 1\text{: answer} \\ 1\text{: justification} \end{cases}$

Section II
Part B — No Calculators

3. p. 22

(a)

$$A = 2\cdot\frac{1}{2}\int_0^{\pi}(2-\cos\theta)^2\,d\theta$$

1: integrand + constant

(b)

Substituting $r = 2-\cos\theta$ into the parametric equations $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ gives

$$\begin{cases} x = (2-\cos\theta)(\cos\theta) = 2\cos\theta - \cos^2\theta \\ y = (2-\cos\theta)(\sin\theta) = 2\sin\theta - (\cos\theta)(\sin\theta) \end{cases}$$

Hence $\begin{cases} \dfrac{dx}{d\theta} = -2\sin\theta + 2(\cos\theta)(\sin\theta) \\ \dfrac{dy}{d\theta} = 2\cos\theta - \cos^2\theta + \sin^2\theta \end{cases}$

$2:\begin{cases} 1: \dfrac{dx}{d\theta} \\ 1: \dfrac{dy}{d\theta} \end{cases}$

(c) $$\frac{dy}{dx} = \frac{2\cos\theta - \cos^2\theta + \sin^2\theta}{-2\sin\theta + 2(\cos\theta)(\sin\theta)}$$

2 : Answer

(d) At $\theta = \frac{\pi}{2}$,

$x = r\cos\theta \Rightarrow x = (2-\cos\theta)(\cos\theta) = (2-\cos\frac{\pi}{2})(\cos\frac{\pi}{2}) = (2-0)(0) = 0$

$y = r\sin\theta \Rightarrow x = (2-\cos\theta)(\sin\theta) = (2-\cos\frac{\pi}{2})(\sin\frac{\pi}{2}) = (2-0)(1) = 2$

Thus, (0, 2) is the point in rectangular coordiantes when $\theta = \frac{\pi}{2}$.

slope: $$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{2\cos\frac{\pi}{2} - \cos^2\frac{\pi}{2} + \sin^2\frac{\pi}{2}}{-2\sin\frac{\pi}{2} + 2(\cos\frac{\pi}{2})(\sin\frac{\pi}{2})} = \frac{2\cdot 0 - 0^2 + 1^2}{-2\cdot 0 + 2(0)(1)} = -\frac{1}{2}$$

Tangent line equation : $y - 2 = -\frac{1}{2}x$

$4:\begin{cases} 2: \text{point} \\ 1: \text{slope} \\ 1: \text{line equation} \end{cases}$

4. p. 23

(a) $\lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{x\to\infty}\left|\frac{x^{2n+2}}{(2n+4)!}\cdot\frac{(2n+2)!}{x^{2n}}\right| = \lim_{x\to\infty}\left|\frac{x^2}{(2n+3)(2n+4)}\right|$

This limit is 0 for any particular real x. Hence, f converges for all x.

3: { 1: sets up ratio test; 1: computes limit; 1: applies ratio test to conclude converges for all x

(b) $g'(x) = 1 - x^2 f(x)$

$$g'(x) = 1 - x^2\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \dots + \frac{(-1)^n x^{2n}}{(2n+2)!} + \dots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots - \frac{(-1)^n x^{2n+2}}{(2n+2)!} + \dots$$

This can also be written $g'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

2: { 1: first 3 terms; 1: general term

(c) We recognize that $g'(x) = \cos x$.

Hence $\cos x = 1 - x^2 f(x)$ and $f(x) = \begin{cases}\frac{1-\cos x}{x^2}, & x \neq 0\\ \frac{1}{2}, & x = 0\end{cases}$.

2: { 1: familar function; 1: answer

(d) Since $g'(x) = \cos x$, we know $g(x) = \sin x + C$.
Since $g(0) = 3$, we determine that $C = 3$ and $g(x) = \sin x + 3$.

2: { 1: antiderivative; 1: answer

5. p. 24

(a) Since $x = \frac{1}{\sqrt{t+1}}$ and $y = \frac{t}{t+1}$

we have $\frac{dx}{dt} = -\frac{1}{2}(t+1)^{-3/2}$ and $\frac{dy}{dt} = \frac{t+1-t}{(t+1)^2}$

$= -\frac{1}{2(t+1)^{3/2}}$ $= \frac{1}{(t+1)^2}$

Thus $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{(t+1)^2} \cdot \frac{2(t+1)^{3/2}}{-1} = -\frac{2}{\sqrt{t+1}}$

> 3: { 1: $\frac{dx}{dt}$; 1: $\frac{dy}{dt}$; 1: $\frac{dy}{dx}$ }

(b) When $t = 3$, we have the point $x = \frac{1}{2}$, $y = \frac{3}{4}$. In addition, the slope is given by $\frac{dy}{dx} = -\frac{2}{2} = -1$. Hence an equation of the tangent line at the point where $t = 3$ is:

$y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$.

This can be rewritten as $x + y = \frac{5}{4}$.

> 2: { 1: slope; 1: equation }

(c)

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{\left(-\frac{1}{2(t+1)^{3/2}}\right)^2 + \left(\frac{1}{(t+1)^2}\right)^2}\, dt = \int_0^1 \sqrt{\frac{1}{4(t+1)^3} + \frac{1}{(t+1)^4}}\, dt$$

> 2: { 1: limits; 1: integrand }

(d) Solve $x = \frac{1}{\sqrt{t+1}}$ for t. ($t \ge 0 \Rightarrow |x| \le 1$ and $y \ge 0$)

$t + 1 = \frac{1}{x^2} \Rightarrow t = \frac{1}{x^2} - 1.$

Substitute this expression for t in terms of x into $y = \frac{t}{t+1}$ to obtain

$$y = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 1 + 1} = 1 - x^2 \quad \text{for } 0 < x \le 1.$$

> 2: { 1: solve for x or y; 1: answer }

6. p. 25

(a) Separate the vaniables and integrate.

$$\frac{dP}{(1200-P)} = k\,dt$$

$$\ln|1200-P| = -kt + C_1$$

$$1200 - P = e^{C_1 - kt} \qquad (0 < P < 1200)$$

$$P = 1200 - Ce^{-kt} \qquad (C = e^{-kt})$$

Since $P(0) = 300$, $C = 900$ and the size of the population is given by

$$P(t) = 1200 - 900e^{-kt}\ .$$

6: { 1: separates variables; 2: antiderivatives; 1: constant of integration; 1: uses initial conditions; 1: solves for P }

(b) $600 = 1200 - 900e^{-kt}$ which means that $e^{-kt} = \frac{2}{3}$. Solving for k, we have

$$k = -\frac{\ln(2/3)}{4} = \frac{1}{4}\ln\frac{3}{2}$$

2: { 1: uses $P(4) = 600$; 1: answer }

(c) Since e^{-kt} tends to zero as t goes to infinity, we have $\lim_{t\to\infty} P(t) = 1200$

1: answer

BC Exam II
Section I
Part A—No Calculators

1. A p. 26

$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^{x^2}(2x\,dx) = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}(e-1)$$

2. C p. 26

$$\lim_{x\to 1}\frac{f(x+1)-f(2)}{x^2-1} = \lim_{x\to 1}\frac{(x+1)^2-1-3}{x^2-1} = \lim_{x\to 1}\frac{x^2+2x+1-1-3}{x^2-1}$$

$$= \lim_{x\to 1}\frac{(x+3)(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x\to 1}\frac{(x+3)}{(x+1)} = 2$$

3. A p. 27

Using implicit differentiation on $\cos x = e^y$, we obtain $-\sin x = e^y\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\sin x}{e^y} = -\frac{\sin x}{\cos x} = -\tan x$$

4. A p. 27

$$y = \text{Arcsin}\left(e^{2x}\right) \quad\Rightarrow\quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(e^{2x}\right)^2}}2e^{2x} = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

5. C p. 27

f is a composite function so use the chain rule when applying the Fundamental Theorem.

$$f(x) = \int_2^{2x}\frac{1}{\sqrt{t^3+1}}\,dt \quad\Rightarrow\quad f'(x) = \frac{1}{\sqrt{(2x)^3+1}}\cdot 2 = \frac{2}{\sqrt{8x^3+1}}$$

Then $f'(1) = \frac{2}{\sqrt{8+1}} = \frac{2}{3}$.

6. C p. 28

$r = 3\csc\theta = \frac{3}{\sin\theta} \quad\Rightarrow\quad r\sin\theta = 3.$

Since in polar coordinates, $y = r\sin\theta$, we have $r = 3\csc\theta$ equivalent to $y = 3$.

7. A p. 28

g is a triple composite and its derivative requires a careful application of the Chain Rule.

$g(x) = \tan^2(e^x) = [\tan(e^x)]^2$

Then $g'(x) = 2[\tan e^x](\sec^2 e^x)(e^x) = 2e^x\tan(e^x)\sec^2(e^x)$

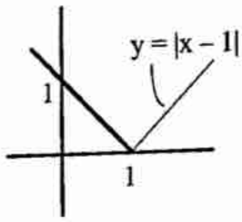
8. D p. 28

Solution I. $\int_0^1 \sqrt{x^2 - 2x + 1}\,dx = \int_0^1 \sqrt{(x-1)^2}\,dx$

On the interval [0, 1], $|x - 1| = 1 - x$.

Therefore $\int_0^1 |x - 1|\,dx = \int_0^1 (1 - x)\,dx = x - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$.

Solution II. As before,
$\sqrt{x^2 - 2x + 1} = |x - 1|$.
Considering the graph of
$y = |x - 1|$, we see that the area under
the graph on the interval [0,1] is $\frac{1}{2}$.



9. B p. 29

$f(x) = \int_0^x \frac{1 + t}{e^t}\,dt$ $f(0) = 0$

$f'(x) = (1 + x)e^{-x}$ $f'(0) = 1$

$f''(x) = e^{-x} + (1 + x)(-e^{-x}) = -xe^{-x}$ $f''(0) = 0$

$f'''(x) = -(e^{-x} + x(-e^{-x})) = -e^{-x} + xe^{-x}$ $f'''(0) = -1$

$f(x) = 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2!} + (-1) \cdot \frac{x^3}{3!} + ... = x - \frac{x^3}{6} + ...$ The answer is (B).

10. D p. 29

Since F' and G' exist, then F and G are continuous.
$F'(x) < 0$ ⇒ F is decreasing everywhere.
$G'(x) > 0$ ⇒ G is increasing everywhere.
If $G(x) > F(x)$ for all x, then the graphs of F and G will never intersect.
If, however, $G(a) \le F(a)$ at some point, then F and G intersect once or not at all.

11. A p. 29

The parametric form for the length of arc is $L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

$\begin{cases} x = \sin t \\ y = t \end{cases}$ ⇒ $\begin{cases} \frac{dx}{dt} = \cos t \\ \frac{dy}{dt} = 1 \end{cases}$

Then $L = \int_0^\pi \sqrt{(\cos t)^2 + 1^2}\,dt = \int_0^\pi \sqrt{\cos^2 t + 1}\,dt$.

12. A p. 30

Since f is continuous $f(0) = \lim_{x\to 0} f(0) = \lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$

$$= \lim_{x\to 0} \frac{-2\sin(2x)}{2x}$$

With the indeterminate form $\frac{0}{0}$, apply L 'Hopitals Rule

$$= \lim_{x\to 0} \frac{-2\sin(2x)}{2x}$$

$$= \lim_{x\to 0} \frac{-2[\cos(2x)\cdot 2]}{x} = -2$$

---

13. D p. 30

Differentiate the expression $\ln(x+y) = x^2$ implicitly. (Watch out for the chain rule!)

$$\frac{1}{x+y}\cdot(1+y') = 2x$$

$(1+y') = 2x^2 + 2xy \quad \Rightarrow \quad y' = 2x^2 + 2xy - 1$

At $x = 1$, $\ln(1+y) = 1 \quad \Rightarrow \quad (1+y) = e$ so that $y = e - 1$.

Then $y' = 2(1) + 2(1)(e-1) - 1 = 2e - 1$

---

14. D p. 30

$f(x) = \sin x + e^{-x}$

$f'(x) = \cos x - e^{-x}$ and $f'(1) = 1 - 1 = 0$. There is a horizontal tangent at $x = 0$.

$f''(x) = -\sin x + e^{-x}$ and $f''(0) = 0 + 1 = 1$. The graph is concave up at $x = 0$.

---

15. C p. 31

To determine the radius of convergence, we evaluate $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+2)(x-3)^{k+1}}{(2k+3)\,2^{k+1}}\cdot\frac{(2k+1)\,2^k}{(k+1)(x-3)^k}\right|$$

$$= \left|\frac{x-3}{2}\right|\cdot\lim_{k\to\infty}\frac{(k+2)(2k+1)}{(2k+3)(k+1)}$$

$$= \left|\frac{x-3}{2}\right|\cdot\lim_{k\to\infty}\frac{2k^2+5k+2}{2k^2+5k+3} = \left|\frac{x-3}{2}\right|$$

The series converges if $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$. Thus we must have $\left|\frac{x-3}{2}\right| < 1$.

This implies that $|x-3| < 2$. This gives an interval centered at $x = 3$, with radius 2. The radius of convergence is $r = 2$.

---

16. E p. 31

Since x and y are functions of t, differentiating $x^2y = 2$ with respect to t gives

$$\left(2x \cdot \frac{dx}{dt} \cdot y\right) + \left(x^2 \cdot \frac{dy}{dt}\right) = 0.$$

With $x = -1$ and $\frac{dy}{dt} = 8$, we have $y = 2$ (from $x^2y = 2$) so that

$$2(-1)\frac{dx}{dt}(2) + (-1)^2(8) = 0 \Rightarrow -4\frac{dx}{dt} = -8 \text{ and } \frac{dx}{dt} = 2.$$

17. B p. 31

I. It is possible for the function f to have smaller values at **some** points x in the interval [a,b] than the function g and still produce a larger integral value than g for the integral over [a,b]. **False**

II. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = 3 - 2 = 1$ **True**

III. As a general principle,
$\int_a^b (f(x) \cdot g(x))\,dx$ is **not** equal to $\left(\int_a^b f(x)\,dx\right) \cdot \left(\int_a^b g(x)\,dx\right)$ **False**

18. A p. 32

$$\begin{cases} x = \frac{2}{t} \\ y = \ln t \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = -\frac{2}{t^2} \\ \frac{dy}{dt} = \frac{1}{t} \end{cases} \quad \text{Then } \frac{dy}{dx} = \left(\frac{1}{t}\right) \cdot \left(-\frac{t^2}{2}\right) = -\frac{t}{2}.$$

When $x = 1$, $t = 2$. Then slope $= \frac{dy}{dx} = -\frac{2}{2} = -1$.

19. C p. 32

Separating the variables we obtain $y^{-2}\,dy = x\,dx$.

Integrating gives $-y^{-1} = \frac{x^2}{2} + c$

Then $-\frac{1}{y} = \frac{x^2}{2} + c = \frac{x^2 + 2c}{2}$, so that $y = \frac{-2}{x^2 + C}$.

Thus the correct answer is $\frac{-2}{x^2 + 1}$.

20. C p. 32

$$\frac{x}{x+2} = \frac{x + 2 - 2}{x + 2} = 1 - \frac{2}{x + 2}$$

$$\int \frac{x}{x+2}\,dx = \int \left(1 - \frac{2}{x+2}\right)dx = x - 2\ln|x + 2| + C$$

21. D p. 33

$f(2.2) \approx f(2) + f'(2) \cdot \Delta x$

Since $f(2) = 4$ and $f'(2) = \sqrt{9} = 3$, then with $\Delta x = 0.2$ we have

$f(2.2) \approx 4 + 3(0.2) = 4.6$.

22. A p. 33

We use the disk method.

$$V = \pi \int_m^{2m} \left(\frac{1}{\sqrt{x}}\right)^2 dx = \pi \ln x \Big|_m^{2m} = \pi[\ln(2m) - \ln(m)] = \pi \ln\left(\frac{2m}{m}\right) = \pi \ln 2$$

Thus the volume is independent of m.

---

23. A p. 34

$$\mathbf{p}(t) = \left\langle \sin\left(3t - \frac{\pi}{2}\right), 3t^2 \right\rangle$$

$$\mathbf{v}(t) = \mathbf{p}'(t) = \left\langle 3\cos\left(3t - \frac{\pi}{2}\right), 6t \right\rangle$$

$$\mathbf{v}(t) = \langle 3\cos(\pi), 3\pi \rangle = \langle -3, 3\pi \rangle$$

---

24. E p. 34

I. Diverges: Fails the nth term test; $\lim_{n\to\infty} \frac{n+1}{2n+2} = \frac{1}{2} \neq 0$

II. Converges: p series with $p = 2 > 1$

III. Converges: By Ratio test $|a_{n+1} / a_n| = \frac{n}{n+1} \cdot \frac{2}{3} \to \frac{2}{3} < 1$

---

25. A p. 34

Note that $F'(x) = G(x)$ by the Fundamental Theorem. Then F must be continuous on (a, b), since it is differentiable.

A. Since $F(a) = 3 = F(b)$, then by Rolle's Theorem, there exists at least one x in (a, b) such that $F'(x) = 0$. Hence $G(x) = 0$ for some x in (a, b). — True

B. If $G(x) = 0$ for **all** x in (a, b), then F is constant on (a, b). This need not be the case. — False

C. $G(x) > 0$ for all x in (a, b) implies that F is increasing on this interval. It is not, since $F(a) = F(b)$ and F is continuous. — False

D. This need not be the case. F can take on negative values in (a, b) with an appropriate function G. — False

E. This need not be the case. F *could* be the function $F(x) = 3$. — False

---

26. B p. 35

Use Integration by Parts.

Take $\left\{ \begin{array}{ll} u = x & u' = 1 \\ v' = e^{-x} & v = -e^{-x} \end{array} \right\}$

Then $\int_0^1 x\, e^{-x}\, dx = \left[-x\, e^{-x} + \int e^{-x}\, dx\right]_0^1$

$= \left[-x\, e^{-x} - e^{-x}\right]_0^1$

$= (-e^{-1} - e^{-1}) - (0 - 1)$

$= 1 - \frac{2}{e}$

---

27. E p. 35

Since the average rate of change of f over the interval $[2, 2 + h]$ is $7e^h - 4\cos(2h)$, we know that

$$\frac{f(2 + h) - f(2)}{h} = 7e^h - 4\cos(2h).$$

$\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = f'(2)$, by the definition of the derivative.

Hence $f'(2) = \lim_{h \to 0} (7e^h - 4\cos(2h)) = 7 - 4 = 3.$

---

28. E p. 35

The given slope field indicates a curve that is increasing and concave up.

A. $y = e^{-x}$ is decreasing, not increasing.

B. $y = \sin x$ is periodic, not always increasing.

C. $y = \sqrt{x}$ is concave down, not concave up.

D. $y = \ln x$ is concave down, not concave up.

E. $y = e^{0.5x}$ is both increasing and concave up.

The correct answer is E.

BC Exam II
Section I
Part B—Calculators Permitted

1. C p. 36

$f(x) = (1+x)^{1/3}$, $f'(x) = \frac{1}{3}(1+x)^{-2/3}$, $f''(x) = -\frac{2}{9}(1+x)^{-5/3}$

$f(0) = 1$, $f'(0) = \frac{1}{3}$, $f''(0) = -\frac{2}{9}$; $a_0 = 1$, $a_1 = \frac{1}{3}$, $a_2 = -\frac{1}{9}$

$T_2(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$

---

2. C p. 36

The critical numbers for f are $x = 1, 4$.

I. $f'(2) > 0$. Hence f is increasing at $x = 2$. **False**

II. $f'(x)$ does not change sign at $x = 1$. Hence there is no relative extreme value at $x = 1$. **False**

III. $f''(x) = 2(x-1)(4-x) - (x-1)^2 = (x-1)(9-3x)$.
Thus $f''(x) > 0$ for $1 < x < 3$. **True**

---

3. D p. 37

$$y = \frac{4x}{1+x^3} \qquad \Rightarrow \qquad y' = \frac{(1+x^3)4 - 4x(3x^2)}{(1+x^3)^2}.$$

Hence $y'(1) = \dfrac{2\cdot 4 - 4\cdot 3}{4} = -1$.

Thus, the tangent line is $y - 2 = -(x-1)$ or $y = -x + 3$.

The intercepts of the tangent line are $(0, 3)$ and $(3, 0)$.

Then the area of the triangle is $\frac{1}{2}\cdot 3\cdot 3 = 4.5$.

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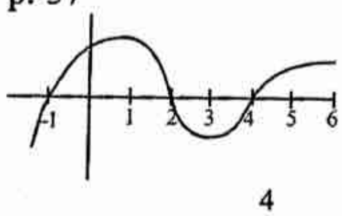
4. C p. 37



I. $f'(4) - f'(2) = \int_2^4 f''(x)\,dx$. Since $\int_2^4 f''(x)\,dx < 0$, we have $f'(4) < f'(2)$ **False**

II. Since the area of the region between the x-axis and the graph of f'' on the interval $[-1,2]$ is greater than the area between the axis and the graph of f'' on the interval $[2,4]$, $\int_{-1}^4 f''(x)\,dx > 0$.

Thus, $f'(4) - f'(-1) = \int_{-1}^4 f''(x)\,dx > 0$ implies $f'(4) > f'(-1)$ **True**

III. See the answer to II. **True**

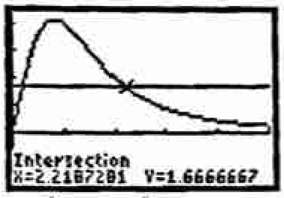
---

5. C p. 37

Ave. velocity = $\dfrac{s(4) - s(1)}{4 - 1} = \dfrac{8 - 3}{4 - 1} = \dfrac{5}{3}$

Inst. velocity = $s'(t) = \dfrac{(t^2 + 2)(18t) - (9t^2)(2t)}{(t^2 + 2)^2} = \dfrac{36t}{(t^2 + 2)^2}$

Use a calculator to find the intersection in the interval [1,4] of the functions $Y1 = 5/3$ and $Y2 = 36x/(x^2 + 2)^2$.
The intersection is at $x \approx 2.219$.



$0 \le x \le 5$
$-1.8 \le y \le 4.4$

---

6. C p. 38

$f(x) = \dfrac{e^x}{x + 1} \quad \Rightarrow \quad f'(x) = \dfrac{(x + 1)e^x - e^x}{(x + 1)^2} = \dfrac{x e^x}{(x + 1)^2}$

$g(x) = \dfrac{x}{x + 1} \quad \Rightarrow \quad g'(x) = \dfrac{(x + 1) - x}{(x + 1)^2} = \dfrac{1}{(x + 1)^2}$

$f'(x) = g'(x)$ if the numerators are equal.

We solve the equation $x e^x = 1$ using the graphing calculator.
We obtain $x \approx 0.567$.

---

7. E p. 38

I. $f(1) = 1$	The value of the function f at $x = 1$.	
$f'(3) < 0$	The slope of the tangent to f at $x = 3$.	**True**
II. $\int_1^2 f(x)\,dx > 1.5$	The area under the curve on $1 \le x \le 2$.	
$f'(3.5) = 0$	The slope of the tangent at $x = 3.5$.	**True**
III. $\lim\limits_{h \to 0} \dfrac{f(2 + h) - f(2)}{h} = f'(2) \approx 1$		
$\dfrac{f(2.5) - f(2)}{2.5 - 2} \approx \dfrac{2.5 - 2.3}{2.5 - 2} = 0.4$		**True**

---

8. B p. 38

I. $\displaystyle\int_0^\infty \frac{1}{1 + x^2}\,dx = \lim_{a \to \infty} \int_0^a \frac{1}{1 + x^2}\,dx = \lim_{a \to \infty} \text{Arctan}\,x \Big|_0^a = \lim_{a \to \infty} (\text{Arctan}\,a - \text{Arctan}\,0) = \frac{\pi}{2}$

Converges

II. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a \to \infty} \int_1^a x^{-2}\,dx = \lim_{a \to \infty} -\frac{1}{x} \Big|_0^a = \lim_{a \to \infty} \left(-\frac{1}{a} + 1\right) = 1$

Converges

III. $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x}\,dx = \lim_{a \to 0^+} \ln x \Big|_a^1 = \lim_{a \to 0^+} (\ln 1 - \ln a) = \lim_{a \to 0^+} (-\ln a) = \infty$

Diverges

---

9. E p. 39

The rate of change of $\frac{dP}{dt}$ is

$$\frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}\left(9P - 6P^2\right) = 9 - 12P, \quad 0 \le P \le 1$$

The critical number occurs at $P = \frac{3}{4}$. Since this derivative is greater than zero to the left of the critical number and less than zero to the right, there is is a maximum when $P = \frac{3}{4}$. So 75% of the population has heard the rumor when it is spreading the fastest.

---

10. A p. 39

We separate the variables of $\frac{dy}{dx} = \frac{x\sin(x^2)}{y}$ to obtain $y\,dy = x\sin(x^2)\,dx$.

Integrating both sides of this equation yields $\frac{y^2}{2} = -\frac{1}{2}\cos(x^2) + k$.

Multiplying by 2 gives $y^2 = -\cos(x^2) + C$.
At $(0, 1)$, we have $1 = -1 + C$, so that $C = 2$.
Then $y^2 = 2 - \cos(x^2)$; hence one solution is $y = \sqrt{2 - \cos(x^2)}$.

---

11. D p. 39

If we consider the function $F(x) = \int_0^x (t^2 + 7)^{2/3}\,dt$, then $g(x) = F(x^2)$.

Hence by the Chain Rule, $g'(x) = F'(x^2) \cdot (2x)$.
By the Fundamental Theorem, $F'(x) = (t^2 + 7)^{2/3}$, so that $F'(x^2) = (x^4 + 7)^{2/3}$.
Then $g'(x) = (x^4 + 7)^{2/3} \cdot (2x)$.
Thus $g''(x) = (x^4 + 7)^{2/3} \cdot (2) + (2x) \cdot \frac{2}{3}(x^4 + 7)^{-1/3} \cdot (4x^3)$.

This gives $g''(1) = 4 \cdot 2 + 2 \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 4 = \frac{32}{3}$.

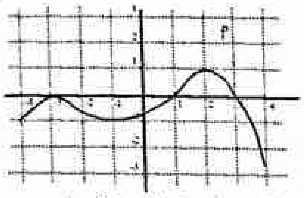
---

12. B p. 40

The given Maclaurin series is that for the function $f(x) = e^{2x}$. We use the calculator to find the intersection of the curves $Y_1 = e^{2x}$ and $Y_1 = 2 - x^3$
The intersection is at x = 0.337

---

13. D p. 40



graph of the derivative of f

I. $f'(-3) = 0$, so f does have a horizontal tangent at $x = -3$.
$f'(2) \approx 1$, so f does not have a horizontal tangent at $x = 2$. **False**

II. $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$.
f is decreasing to the left of $x = 1$ and is increasing to the right. **True**

III. $f'(x)$ is increasing to the left of $x = -3$, so the graph of f is concave up there.
$f'(x)$ is decreasing to the right of $x = -3$, so the graph of f is concave down there.
The concavity of the graph of f changes at $x = -3$. **True**

---

14. B p. 41

We use the washer method.

$$V = \pi \int_0^{\ln 3} \left[(e^{x/2})^2 - 1^2 \right] dx = \pi \int_0^{\ln 3} (e^x - 1)\, dx \approx 2.832$$

---

15. C p. 41

The Lagrange error is $|R_n(x)| = \frac{M}{(n+1)!}|x-a|^{n+1}$

When n = 2, a = 1, x = 1.4 and M = 10 we have

$|R_2(1.4)| = \frac{10}{3!}|1.4-1|^3 = 0.107$

Recall that M is the upper bound for all values of the (n+1) derivative on the given interval , [a, x] in this case.

---

16. E p. 42

I. This is the additive property of integrals. **True**

II. Since f is differentiable on [a,b], it is continuous as well. Then the Mean Value Theorem holds. **True**

III. Since f is differentiable, it is continuous at c. Hence $\lim_{x \to c} f(x) = f(c)$. **True**

---

17. B p. 42

From similar triangles, $\frac{R}{h} = \frac{6}{4} = \frac{3}{2}$. Thus at any particular depth, $R = \frac{3}{2}h$. In the formula for the volume of a cone, we then have $V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3$. Differentiating with respect to time t, we have $\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$.

We are given that $\frac{dV}{dt} = 10$, and are to determine $\frac{dh}{dt}$ when $h = 2$.

This produces $10 = \frac{9}{4}\pi \cdot 4 \frac{dh}{dt}$. Hence $\frac{dh}{dt} = \frac{10}{9\pi} \approx 0.354$ ft/min.

Exam II
Section II
Part A — Calculators Permitted

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1. p. 44

(a) To find the velocity, the derivative of s, we apply the Fundamental Theorem to

$$s(t) = \int_1^t [1 - x\cos x - (\ln x)(\sin x)]\,dx \quad .$$

This gives $s'(t) = v(t) = 1 - t\cos t - (\ln t)(\sin t)$.

1: formula

(b) Solving $\frac{d}{dt}|s'(t)| = 0$, for $1 \le t \le 8$, we have t=1.234 or t = 3.731 or t = 6.700. Evaluating the speed at the interior critical points:

$$|s'(1.234)| = 0.394 \quad \text{and} \quad |s'(3.731)| = 4.833 \quad \text{and} \quad |s'(6.700)| = 5.896$$

At the endpoints:

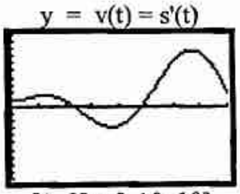
$$|s'(1)| = 0.460 \text{ and } |s'(8)| = 0.107.$$

Thus, the particle reaches its maximum speed of 5.896 at t = 6.70.

4: { 1: Sets $\frac{d}{dt}|s'(t)| = 0$; 1: interior critical pts; 1: evaluation at critical pts / endpts; 1: answer }

(c) The particle is moving to the left when its velocity is negative. We graph v(t) on its domain $1 \le t \le 8$. The zeros of the velocity function are at t = 5.204 and t = 7.987. We see from the graph that the velocity is negative-valued between its zeros. Hence the particle is moving left when $5.204 < t < 7.987$.

y = v(t) = s'(t)



[1, 8] x [–10, 10]

2: { 1: v(t) < 0; 1: answer }

(d) Since the particle moves both left and right, we must integrate $|v(t)|$ to determine the total distance traveled.

$$\text{Total distance} = \int_1^8 |1 - x\cos x - (\ln x)(\sin x)|\,dx \approx 21.461 .$$

2: { 1: integral; 1: answer }

2. p. 45

(a) Averaging the slopes of the secants on each side of $h = 22$ gives

$$\frac{1}{2}\left[\frac{1.3 - 1.5}{26 - 22} + \frac{1.5 - 1.6}{22 - 18}\right] = -0.0375 \text{ feet per foot.}$$

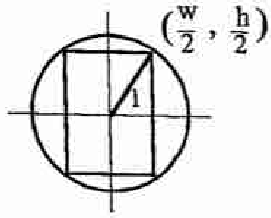
2: { 1: difference quotient; 1: answer (with units) }

(b) From $h = 14$ to $h = 30$, there are four subintervals, each of width 4. We must also convert the given diameters to radii in order to approximate $V = \int_{14}^{30} \pi r^2 \, dx$

$$T_4 = \frac{4}{2}\pi\left[\left(\frac{1.8}{2}\right)^2 + 2\left(\frac{1.6}{2}\right)^2 + 2\left(\frac{1.5}{2}\right)^2 + 2\left(\frac{1.3}{2}\right)^2 + \left(\frac{1.2}{2}\right)^2\right] = 27.772 ft^3$$

3: { 2: trapezoid methods; 1: answer (with units) }

(c) From $h = 2$ to $h = 8$, cross-sections are all circular with diameter 2. The strength of the beam is given by $S = kwh^2$ where $\left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2 = 1$.



Thus $h^2 = 4 - w^2$ and $S = kw(4 - w^2) = 4kw - kw^3$

Then $S' = 4k - 3kw^2$. Hence $S' = 0$ when $w = 1.155$. The candidates for maximum are the critical number $w = 1.155$ and the endpoints $w = 0$ and $w = 2$. At the end points $w = 2$ and $w = 0$, $S = 0$. Since $S(1.155) > 0$, the maximum is at $w = 1.155$ ft.and and $h = 1.633$ ft.

4: { 1: $s = kwh^2$; 1: solving $\frac{ds}{dw} = 0$; 1: evaluation at critical pts and endpts; 1: answer }

Exam II
Section II
Part B — No Calculators

3. p. 46

(a) $$\text{Ave value} = \frac{1}{3 - 1}\int_1^3 \left(x + \frac{2}{x}\right) dx = \frac{1}{2}\left[\frac{x^2}{2} + 2\ln x\right]_1^3 = 2 + \ln 3$$

3: { 1: limits / constant; 1: integrand; 1: answer }

(b) Since both k and x are positive on [1, 3], then $g(x) = x + \frac{k}{x} > x = f(x)$.

$$V = \pi\int_1^3 [(x + \tfrac{k}{x})^2 - x^2]\, dx) \text{ (washer method)}$$

$$= \pi\int_1^3 [2k + \tfrac{k^2}{x^2}]\, dx = \pi\left[2kx - \frac{k^2}{x}\right]_1^3 = \frac{\pi}{3}(12k + 2k^2)$$

3: { 1: limits / constant; 1: integrand; 1: answer }

(c) Again we use the washer method.

The outer radius of a typical cross section of the solid is $g(x) + 2 = x + \frac{k}{x} + 2$.
The inner radius of a typical cross section of the solid is $f(x) + 2 = x + 2$.
Then the volume of the solid is

$$\pi \int_1^3 \left[(x + \frac{k}{x} + 2)^2 - (x + 2)^2\right] dx.$$

3: { 1: limits / constant; 1: integrand; 1: answer }

4. p. 47

(a) Approximate the slope of f' on each side of $x = 2$.

$$\frac{f'(2) - f'(1)}{2 - 1} = \frac{19 - 21}{1} = -2, \quad \frac{f'(3) - f'(2)}{3 - 2} = \frac{15 - 19}{1} = -4,$$

$$\frac{f'(3) - f'(1)}{3 - 1} = \frac{15 - 21}{3 - 1} = -3.$$ Any of these are appropriate.

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(b) $$\int_0^2 x f'(x^2)\,dx = \frac{1}{2}\int_0^2 f'(x^2)\,(2x\,dx)$$

Note that $2x = D_x(x^2)$, so that $(2x)\,f'(x^2)$ is the derivative of $f(x^2)$.

Thus: $$\int_0^2 x f'(x^2)\,dx = \frac{1}{2}\int_0^2 (2x)\,f'(x^2)\,dx = \frac{1}{2}\Big[\, f(x^2) \,\Big]_0^2$$

$$= \frac{1}{2}[\,f(4) - f(0)\,] = \frac{1}{2}(9 - 1) = 4$$

$3: \begin{cases} 2: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(c) To evaluate $\int_1^3 x f''(x)\,dx$, we use Integration by Parts.

Let $u = x$ $\quad u' = 1$

$v' = f''(x)$ $\quad v = f'(x)$

Then $$\int_1^3 x f''(x)\,dx = x f'(x) - \int 1 \cdot f'(x)\,dx \Big|_1^3$$

$$= x f'(x) - f(x)\Big|_1^3 = (3 \cdot f'(3) - f(3)) - (1 \cdot f'(1) - f(1))$$

$$= (3 \cdot 15 - 8) - (21 - 17) = 37 - 4 = 33$$

$4: \begin{cases} 3: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

5. p. 48

(a) $$f(x) = xe^{-kx} \Rightarrow f'(x) = e^{-kx} - kx\,e^{-kx} = e^{-kx}(1 - kx) = \frac{1 - kx}{e^{kx}}$$

Critical numbers occur if this derivative is either 0 or undefined. Since $e^{kx} > 0$ for all x, f' changes from + to − at $x = \frac{1}{k}$, so that is the only critical number.

$2: \begin{cases} 1: \text{sets } f'(x) = 0 \\ 1: \text{critical point} \end{cases}$

(b) $$f''(x) = -k\,e^{-kx} - (ke^{-kx} - k^2 x\,e^{-kx}) = k^2 x\,e^{-kx} - 2k\,e^{-kx} = ke^{-kx}(kx - 2)$$

$$f''\left(\frac{1}{k}\right) = k\,e^{-1}(1 - 2) = -\frac{k}{e} < 0$$

Since the curve is concave down at $x = \frac{1}{k}$, there is a relative maximum at $\left(\frac{1}{k}, \frac{1}{ke}\right)$.

$3: \begin{cases} 1: \text{find } f''(x) \\ 1: \text{evaluate } f''\left(\frac{1}{k}\right) \\ 1: \text{answer with reason} \end{cases}$

(c) $$f''(x) = ke^{-kx}(kx - 2) = \frac{k\,(kx - 2)}{e^{kx}}.$$

$f''(x) > 0$ if and only if $x > \frac{2}{k}$, so the graph is concave up for $x > \frac{2}{k}$.

$2: \begin{cases} 1: \text{sets } f''(x) > 0 \\ 1: \text{answer} \end{cases}$

(d) $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} xe^{-kx} = \lim_{x \to \infty} \frac{x}{e^{kx}}.$$ This takes on the indeterminate form $\frac{0}{0}$.

Hence we use L'Hôpital's Rule. $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{ke^{kx}} = 0.$$

Thus the horizontal asymptote is the line $y = 0$.

$2: \begin{cases} 1: \lim\limits_{x \to \infty} xe^{-kx} \\ 1: \text{answer} \end{cases}$

6. p. 49

(a) $f(x) = \ln(x+1) \Rightarrow f'(x) = \frac{1}{x+1} = (x+1)^{-1}$

$\Rightarrow f''(x) = -(x+1)^{-2} \Rightarrow f'''(x) = 2(x+1)^{-3}$

Then $f'(0) = 1$ $f''(0) = -1$ $f'''(0) = 2$

2: $\begin{cases} 1: f'(0) \text{ and } f''(0) \\ 1: f'''(0) \end{cases}$

(b) First of all, $f(0) = 0$.

The coefficients of $f^{(n)}(x)$ develop as alternating in sign and growing as factorials.

In general, $f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(x+1)^{-n}$, so $f^{(n)}(0) = (-1)^{n+1}(n-1)!$

Hence $f(x) = 0 + x - \frac{x^2}{2!} + 2\cdot\frac{x^3}{3!} - 6\cdot\frac{x^4}{4!} + \ldots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots$

2: $\begin{cases} 1: \text{first 3 terms} \\ 1: \text{general term} \end{cases}$

(c) *Solution I.* We use the Ratio Test to determine the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}x^n}\right|$$

$$= \lim_{n\to\infty}\left|-x\cdot\frac{n}{n+1}\right| = |x|$$

We have convergence if $|x| < 1$; that is, if $-1 < x < 1$.

Thus the radius of convergence is $R = 1$.

3: $\begin{cases} 1: \text{sets up ratio} \\ 1: \text{limit} \\ 1: \text{applies ratio test} \end{cases}$

Solution II. $f'(x) = \frac{1}{x+1} = \frac{1}{1-(-x)}$.

The power series for f' can be obtained as a geometric series with first term $a = 1$ and common ratio $r = -x$. This geometric series converges if and only if its common ratio is strictly between -1

and 1. This is an interval of radius $R = 1$.

If the power series for f' converges on an interval of radius 1, then so does the power series for f. Hence we have $R = 1$.

(d) $$\int_0^{0.5} f(x)\,dx = \left[\frac{1}{2}x^2 - \frac{x^3}{2\cdot3} + \frac{x^4}{3\cdot4} - \frac{x^5}{4\cdot5} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{n(n+1)} + \cdots\right]_0^{0.5}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{2\cdot3}\left(\frac{1}{2}\right)^3 + \frac{1}{3\cdot4}\left(\frac{1}{2}\right)^4 - \frac{1}{4\cdot5}\left(\frac{1}{2}\right)^5 + \cdots$$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error in using a part of the series as an approximation of the definite integral is less than the first omitted term. Since absolute value of the third term is less than 0.01,

$$\frac{1}{3\cdot4}\left(\frac{1}{2}\right)^4 = \frac{1}{192} < \frac{1}{100},$$

we have, $\int_0^{0.5} f(x)\,dx \approx \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{2\cdot3}\left(\frac{1}{2}\right)^3 = \frac{5}{48}$ with an error no greater than 0.01.

2: $\begin{cases} 1: \text{evaluation} \\ 1: \text{refers to alternating series and indicates the error term is found from first omitted term} \end{cases}$

Exam III
Section I
Part A — No Calculators

1. C p. 50

$x(t) = (\ln t)^2$

Then the velocity is given by $v(t) = x'(t) = 2\,(\ln t)\cdot\frac{1}{t}$.

The maximum value of the velocity will occur at a critical number of this function.

$v'(t) = 2\cdot\frac{1}{t}\cdot\frac{1}{t} + 2\,(\ln t)\,(-\frac{1}{t^2}) = \frac{2}{t^2}(1-\ln t)$

$v'(t) = 0 \quad \Leftrightarrow \quad \ln t = 1 \quad \Leftrightarrow \quad t = e$

2. C p. 50

$\int_0^{\pi} \cos x\,dx = 0$, because on the interval $[0,\pi]$, cosine takes on both positive and negative values in such a way that the definite integral has the value 0.

The only one of the suggested answers that matches this property is $\int_{-\pi/2}^{\pi/2} \sin x\,dx$

3. C p. 51

$\frac{dy}{dx} = \frac{1}{xy}$.

We separate variables to obtain $y\,dy = \frac{1}{x}\,dx$.

Integrating, we obtain $\frac{y^2}{2} = \ln|x| + C,$

or $y^2 = 2\ln|x| + K.$

Using the initial condition $y(1) = 1$ gives $1 = 0 + K$, so $K = 1$.

Since the point $(1,1)$ is on the curve and $x \neq 0$, it follows that $x > 0$ and we don't need the absolute value.

Hence $y^2 = 2\ln x + 1$;

and since the curve contains a point where $y > 0$, we have

$y = \sqrt{2\ln x + 1}$

4. D p. 51

I. $f'(x) > 0$ on the interval $(-3,-2)$, so f is increasing there. **True**

II. f' is decreasing on $(-3,-1)$, so the graph of f is concave down there. **True**

III. f is increasing on $(-3,-2)$, so $f(-3) < f(-2)$, and the maximum value can't be $f(-3)$. **False**

5. D p. 51

$$\int_e^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_e^{b} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_e^{b} (\ln x)^{-2} \cdot \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} -(\ln x)^{-1}\Big|_e^b = \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln e}\right) = 1$$

---

6. E p. 52

A radius of convergence of 1 implies $|x-2| < 1$. Thus the series converges for all x, $1 < x < 3$.

Checking endpoints: $x = 1 \Rightarrow$ convergent alternating harmonic series $\sum_1^{\infty} \frac{(-1)^n}{n}$

$x = 3 \Rightarrow$ divergent harmonic series $\sum_1^{\infty} \frac{1}{n}$

Interval of convergence : $1 \le x < 3$..

---

7. E p. 52

$\frac{dP}{dt} = 2P - 0.01P2 \Rightarrow \frac{dP}{dt} = 2P(1 - \frac{1}{200}P)$

This is an example of the logistic equation.

When P = 0 and again when P = 200, we have $\frac{dP}{dt} = 0$.

Half-way between is when $\frac{dP}{dt}$ is maximum. (I) is True.

If P > 0, then $\frac{dP}{dt} < 0 \Rightarrow$ P is decreasing (II) is True.

The carrying capacity of the population is 200. (III) is True, hence the answer is E.

---

8. C p. 53

If we take $x = \sin\theta$,
$dx = \cos\theta\, d\theta$

and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$.

In addition, we change the limits of integration. When $x = 0$, then $\theta = 0$ and when $x = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$.

Hence the new integral is $\int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta\, d\theta = \int_0^{\pi/6} \sin^2\theta\, d\theta$.

---

9. C p. 53

Since $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, we use L'Hôpital's Rule.

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{6}{2} = 3.$

---

10. C p. 53

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{1 + \frac{x^2}{4}} = \frac{2}{4 + x^2}$$

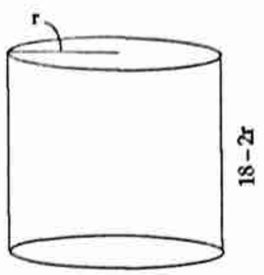
Hence at the point $(2, \frac{\pi}{4})$, the slope of the tangent is $\frac{1}{4}$.

11. E p. 54

$$\frac{dy}{dx} = \frac{3\sin x}{\sec^2 x} = 3\sin x\cos^2 x$$

$$\begin{aligned} \text{Then } y &= \int 3\sin x\cos^2 x\,dx \\ &= -\int 3\cos^2 x\,(-\sin x)\,dx \\ &= -\cos^3 x + C \end{aligned}$$

12. E p. 54



$V = \pi r^2 h$

Since the height of the cylinder is $18 - 2r$, we have:

$V(r) = \pi r^2 (18 - 2r) = 18\pi r^2 - 2\pi r^3$.

Then $V'(r) = 36\pi r - 6\pi r^2$.

Set this equal to 0 to find the critical numbers.

$6\pi r(6 - r) = 0$.

Hence $r = 0$ or 6.

The critical number $r = 0$ gives the volume $V(0) = 0$.

The critical number $r = 6$ gives the volume $V(6) = 36 \cdot 6\pi$.

This is the maximum volume. Hence $r = 6$.

13. C p. 54

$r = 1 + \sin\theta$ is a cardioid.

The area is $\frac{1}{2}\int_0^{2\pi} (1 + \sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta)\,d\theta$ $\quad [\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)]$

$$= \frac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos(2\theta))\,d\theta$$

Because of the periodicity of $\sin\theta$ and $\cos 2\theta$,

$$\int_0^{2\pi} \sin\theta\,d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \cos(2\theta)\,d\theta = 0.$$

Hence the area is: $\frac{1}{2}\int_0^{2\pi} \frac{3}{2}\,d\theta = \frac{1}{2} \cdot \frac{3}{2} \cdot 2\pi = \frac{3\pi}{2}$.

14. A p. 55

$a(t) = 1 + e^{-t}$

$v(t) = \int a(t)\,dt = t - e^{-t} + C$

$v(0) = -2 \Rightarrow -2 = 0 - e^0 + C \Rightarrow C = -1$

Thus $v(t) = t - e^{-t} - 1$

$s(t) = \int v(t)\,dt = \frac{t^2}{2} + e^{-t} - t + D$

$s(0) = 3 \Rightarrow 3 = 0 + e^0 - 0 + D \Rightarrow D = 2$

Hence $s(t) = \frac{t^2}{2} + e^{-t} - t + 2$

---

15. E p. 55

$y = \tan x$

$y' = \sec^2 x$, so $(y')^2 = \sec^4 x$. Hence the arc length is $\int_a^b \sqrt{1 + \sec^4 x}\,dx$.

---

16. D p. 56

$v = \sin(u^2 - 1)$ and $u = \sqrt{x^2 + 1}$.

Solution I. $u^2 = x^2 + 1$, so $u^2 - 1 = x^2$.

Then $v = \sin(u^2 - 1) \Rightarrow v = \sin(x^2)$

$\Rightarrow \frac{dv}{dx} = 2x\cos(x^2)$

Solution II. $\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx} = [2u\cos(u^2 - 1)] \cdot \frac{2x}{\sqrt{x^2 + 1}}$

Now we substitute for u:

$\frac{dv}{dx} = [2\sqrt{x^2 + 1}\cos(x^2)] \cdot \frac{x}{\sqrt{x^2 + 1}} = 2x\cos(x^2)$

---

17. E p. 56

A, B, C, and D are all true because the definition of continuity of the function f at $x = c$ requires that:

i. $f(c)$ exist;

ii. $\lim_{x \to c} f(x)$ exist;

iii. $\lim_{x \to c} f(x) = f(c)$.

Hence answer E is the one that could be false. For example, the absolute value function, $f(x) = |x|$ is continuous at $x = 0$, but $f'(0)$ does not exist.

---

18. D p. 56

$$\int_{1/2}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} \int_{1/2}^{b} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{b\to 1^-} \text{Arcsin } x\Big]_{1/2}^{b}$$

$$= \lim_{b\to 1^-} \left(\text{Arcsin } b - \text{Arcsin } \frac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

---

19. D p. 57

Consider the function $F(x) = 3x^2 - x^3$.
The relative maximum and minimum values of F allow us to determine how much vertical shift is allowed for F if we are to obtain three real zeros for f.

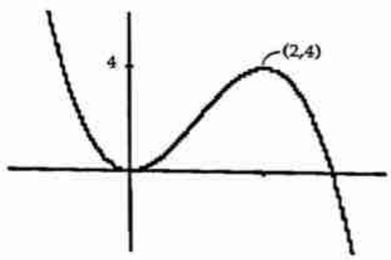
$F'(x) = 6x - 3x^2 = 3x(2 - x)$. The critical numbers for F are 0 and 2. $F(0) = 0$ while $F(2) = 4$.

The graph of F is shown to the right.

Note that $f(x) = F(x) + h$.

If we move F down some, but less than 4 units, then the resulting function will have three real zeros.

Hence $-4 < h < 0$.



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20. C p. 57

Given $f'''(0) = 4$. Then the coefficient of x^3 in the Taylor series for f about $x = 0$ is $\frac{f'''(0)}{6} = \frac{4}{6} = \frac{2}{3}$.

---

21. C p. 57

The depth increases rapidly at first; then it increases very slowly for a while; and finally it increases more rapidly. The container must be narrow at the top and bottom, and quite wide in the middle.

---

22. D p. 58

$\frac{dS}{dt} = 300t + t^{1/2} + t^{3/2}$. Then the accumulated sales over four days is:

$$\int_0^4 (300t + t^{1/2} + t^{3/2})\,dt = \left[150\,t^2 + \frac{2}{3}t^{3/2} + \frac{2}{5}\,t^{5/2}\right]_0^4$$

$$= 150 \cdot 16 + \frac{2}{3} \cdot 8 + \frac{2}{5} \cdot 32 = 2400 + \frac{16}{3} + \frac{64}{5}$$

$$\approx 2400 + 5.3 + 12.8 \approx 2418$$

---

23. C p. 58

$$F(x) = \int_{\pi/2}^{x} 4t\sin\left(\frac{t}{3}\right)dt \quad \Rightarrow \quad F'(x) = 4x\sin\left(\frac{x}{3}\right).$$

Hence $F'\left(\frac{\pi}{2}\right) = 4\cdot\frac{\pi}{2}\cdot\sin\frac{\pi}{6} = \pi$.

At $x = \frac{\pi}{2}$, $F\left(\frac{\pi}{2}\right) = \int_{\pi/2}^{\pi/2} 4t\sin\left(\frac{t}{3}\right)dt = 0$ because the length of the interval is 0.

Then the equation of the tangent line at the point $\left(\frac{\pi}{2}, 0\right)$ is

$$y - 0 = \pi\left(x - \frac{\pi}{2}\right) \quad \Rightarrow \quad y = \pi x - \frac{\pi^2}{2} \quad \Rightarrow \quad 2y = 2\pi x - \pi^2$$

---

24. B p. 58

$$\int_0^k \frac{\sec^2 x}{1 + \tan x}dx = \ln\left|1 + \tan x\right|\Big|_0^k = \ln\left|1 + \tan k\right| = \ln 2.$$

This implies that $\left|1 + \tan k\right| = 2$, so that $1 + \tan k = \pm 2$.
Hence $\tan k = 1$ or $\tan k = -3$. But $\tan k = -3$ creates a divergent improper integral, since $1 + \tan x = 0$ on the interval $[\text{Arctan}(-3), 0]$.
Thus $\tan k = 1$, so $k = \frac{\pi}{4}$.

---

25. C p. 59

Since the rate of change of the depth is 4 times the rate of change of the radius, we know that $\frac{dh}{dt} = 4\frac{dr}{dt}$. Hence $h = 4r + C$. But in a cone, when $r = 0$, $h = 0$ as well, so $C = 0$. Thus $h = 4r$.

$$V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad V = \frac{1}{3}\pi r^2(4r) = \frac{4}{3}\pi r^3.$$

Then $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Since $\frac{dV}{dt} = 16$, we have $16 = 4\pi r^2\frac{dr}{dt}$,

so $\frac{dr}{dt} = \frac{4}{\pi r^2}$.

The area of the exposed surface is $A = \pi r^2$.
Then $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r\left[\frac{4}{\pi r^2}\right] = \frac{8}{r}$.

When the radius is 2, $\frac{dA}{dt} = 4$.

---

26. D p. 59

$$\left.\begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array}\right\} \Rightarrow y(\tfrac{1}{2}) \approx 0 + 1\cdot(\tfrac{1}{2}) = \tfrac{1}{2}$$

$$\left.\begin{array}{l} y(\tfrac{1}{2}) = \tfrac{1}{2} \\ y'(\tfrac{1}{2}) = \dfrac{1}{1+\tfrac{1}{2}} = \tfrac{2}{3} \end{array}\right\} \Rightarrow y(1) \approx \tfrac{1}{2} + \tfrac{2}{3}\cdot\tfrac{1}{2} = \tfrac{5}{6}$$

---

27. D p. 60

The function appears to be an "up-side down" exponential with a horizontal asymptote (as $x \to \infty$) at $y = 2$. Hence it looks like $y = 2 - e^{-x}$.

---

28. A p. 60

Use Integration by Parts.

$$\text{Take} \left\{\begin{array}{ll} u = x & u' = 1 \\ v' = e^{2x} & v = \tfrac{1}{2}e^{2x} \end{array}\right\}$$

$$\text{Then } \int x\,e^{2x}\,dx = \tfrac{1}{2}x\,e^{2x} - \tfrac{1}{2}\int e^{2x}\,dx$$

$$= \tfrac{1}{2}x\,e^{2x} - \tfrac{1}{4}e^{2x} + C = \tfrac{1}{4}e^{2x}(2x-1) + C$$

Exam III
Section I
Part B — Calculators Permitted

1. A p. 61

$s(t) = (t-2)^3(t-6)$

$$\begin{aligned} s'(t) &= 3(t-2)^2(t-6) + (t-2)^3 \\ &= (t-2)^2[3(t-6) + (t-2)] \\ &= (t-2)^2(4t-20) \\ &= 4(t-2)^2(t-5) \end{aligned}$$

I. $s'(t) > 0$ for $t > 5$ — **True**

II. $s'(6) > 0$ — **False**

III. $s'(t) < 0$ if $t < 5$, except at $t = 2$
Hence the particle is moving to the left both before and after $t = 2$. — **False**

2. B p. 61

Since $f(x) = \int_0^x \sin(t^2)\,dt$, then the rate of change of the function f is given by $f'(x) = \sin(x^2)$.

The average value of this rate of change on the interval [1,3] is

$$\frac{1}{2}\int_1^3 \sin(x^2)\,dx \approx 0.232\,.$$

3. D p. 62

$$\int \frac{1}{\sqrt{x}\left(1-\sqrt{x}\right)}\,dx = -2\int \frac{1}{1-\sqrt{x}} \cdot \frac{-1}{2\sqrt{x}}\,dx = -2\ln\left|1 - \sqrt{x}\right| + C$$

4. D p. 62

We use the washer method. The outer radius is $R = \ln(x+1) + 1$; the inner radius is $r = 1$.

$$V = \pi\int_0^e \left([\ln(x+1) + 1]^2 - 1^2\right)dx$$

Use a calculator to evaluate this definite integral. The result is $V = 20.146$.

5. C p. 62

Let $F(h) = \int_1^{1+h} \sqrt{x^3 + 8}\,dx$. Then $\lim_{h\to 0} F(h) = 0$. Hence the given limit has the indeterminate form $\frac{0}{0}$. This is an appropriate time to use L'Hôpital's Rule.

The derivative of F(h), with respect to h, is $F'(h) = \sqrt{(1+h)^3 + 8}$.
Hence

$$\lim_{h\to 0}\frac{\int_1^{1+h}\sqrt{x^3+8}\,dx}{h} = \lim_{h\to 0}\frac{\sqrt{(1+h)^3+8}}{1} = 3.$$

6. B p. 63

$h(x) = g(x^2)$. Then by the Chain Rule, $h'(x) = g'(x^2) \cdot 2x$, so $h'(2) = g'(4) \cdot 4$.
We estimate from the graph that $g'(4) \approx -1$. Then $h'(2) \approx -4$.

7. A p. 63

The Lagrange bound on the error in this 4th degree Taylor polynomial is $\frac{M}{(n+1)!}|x-a|^{n+1}$.

At $x = 2.9$, $a = 2$, $n = 4$ and $M = 0.8$, we have $\frac{0.08}{5!}|2.9-2|^5 \approx 0.004$

8. D p. 63

I. Since $f'(x) < 0$ on $(-2,-1)$, f is decreasing on $(-2,-1)$. **False**

II. Since $f'(0)$ exists, f is continuous at $x = 0$. **True**

III. $f''(x) < 0$ if $-4 < x < -2$ since $f'(x)$ is decreasing there.
$f''(x) > 0$ if $-2 < x < 1$ since $f'(x)$ is increasing there.
Hence the concavity of the graph of f changes at $x = -2$. **True**

9. B p. 64

$$\begin{cases} x = e^t \\ y = 2e^{-t} \end{cases}$$

Solution I. When $t = \ln 2$, we have the point $(e^{\ln 2}, 2e^{-\ln 2}) = (2,1)$.

The slope of the curve is $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2e^{-t}}{e^t}$

When $t = \ln 2$, $\frac{dy}{dx} = \frac{-2e^{-\ln 2}}{e^{\ln 2}} = -\frac{1}{2}$

Then the equation of the desired tangent is:

$y - 1 = -\frac{1}{2}(x-2) \Rightarrow 2y - 2 = -x + 2 \Rightarrow x + 2y - 4 = 0$

Solution II. We eliminate the parameter to obtain $y = \frac{2}{x}$. When $t = \ln 2$, $x = 2$, so $y = 1$.
Then $\frac{dy}{dx} = -\frac{2}{x^2}$, so when $x = 2$, $\frac{dy}{dx} = -\frac{1}{2}$. The equation is obtained as before.

Solution III. First we obtain the point desired point $(2,1)$ as before. Then we note that only proposed answers (B) and (E) contain this point.
The slope of equation (B) is $m = -\frac{1}{2}$. The slope of equation (E) is $m = -2$
.Then graph the given curve (either with the non-parametric $y = \frac{2}{x}$ or with the given parametric equations) with a window that has equal units. It is clear that the slope of the curve at $(2,1)$ is <u>much</u> closer to $-\frac{1}{2}$ than to 2.

10. D p. 64

$x^2 - y^2 = 25$

Differentiating implicitly, we obtain $2x - 2y\frac{dy}{dx} = 0$. Then $\frac{dy}{dx} = \frac{x}{y}$.

By the Quotient Rule: $\frac{d^2y}{dx^2} = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2} = \frac{y - \frac{x^2}{y}}{y^2} = \frac{y^2 - x^2}{y^3}$.

Since the curve is $x^2 - y^2 = 25$, the numerator of this second derivative has the value -25. Hence $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

---

11. C p. 65

I. This is a "p-series" with $p = 2$. Convergent

II. This is the alternating harmonic series. Convergent

III. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n^2} = \infty$. $\lim_{n\to\infty} a_n \neq 0$. Divergent

---

12. C p. 65

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \frac{x^2+1}{x^2} \quad \Rightarrow \quad g'(x) = \frac{x^2+1}{x^2}$$

$$\Rightarrow \quad g'(x) = 1 + \frac{1}{x^2}$$

$$\Rightarrow \quad g(x) = x - \frac{1}{x} + C = \frac{x^2-1}{x} + C$$

---

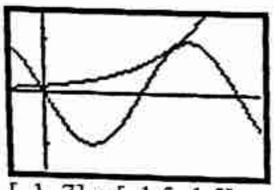
13. A p. 65

$$\begin{cases} x_1 = \cos t \\ x_2 = e^{(t-3)/2} - .75 \end{cases} \quad \text{for } 0 \le t \le 2\pi.$$

Then the velocities of the two particles are given by:

$$\begin{cases} v_1 = -\sin t \\ v_2 = \frac{1}{2}e^{(t-3)/2} \end{cases}$$

Graph these velocities on a viewing window that includes the domain $0 \le t \le 2\pi$.



$[-1, 7] \times [-1.5, 1.5]$

The first intersection shown in the viewing window occurs for a number $x < 0$ (i.e., out of the domain). Hence we look more closely at the region where the curves might intersect toward the upper right of this graph. When we zoom in on this portion of the graph, we find that the velocity curves do not intersect at all in the given domain.

---

14. C p. 66

Parts of the rectangles are above the curves for B and C.

The trapezoids are all on or above the curves for A, C, and E.

Hence, the answer is C.

---

15. C p. 66

To determine the radius of convergence, we evaluate $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\,x}{(n+1)^{n+1}}\cdot\frac{n^n}{1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{(n+1)^n}\cdot\frac{n^n}{1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{\left(\frac{n+1}{n}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{x}{\left(1+\frac{1}{n}\right)^n}\right| = \frac{|x|}{e}$$

The last step is based on the important limit: $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$.

The series converges if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$.

Hence we have convergence if $\dfrac{|x|}{e} < 1$.

That happens if $|x| < e$.

This is an interval centered at the origin and having a radius of e.
Thus the radius of convergence of the series is $R = e$.

16. D p. 67

I. Since f is concave down, f' decreasing. Thus, $f'(1.5) > f'(1.7)$ **False**

II. The average rate of change of f on the intervals [1.3,1.5] and [1.5,1.7] is 15 and 10 respectivelly. Then $f'(1.5)$ must lie between numbers 10 and 15.. **True**

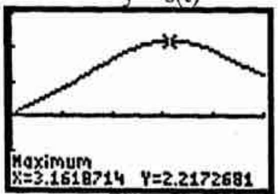
III. Since f is concave down on the interval, $f''(1.7) < 0$. However, $f(1.7) = 23$ **True**

---

17. E p. 67

With the acceleration vector $\mathbf{a} = \langle 2, e^{-t} \rangle$, we antidifferentiate to obtain the velocity vector $\mathbf{v} = \langle 2t + A, -e^{-t} + B \rangle$.
The particle is at rest when $t = 0$, so $\mathbf{v}(0) = \langle 0,0 \rangle$.
Since $\mathbf{v}(0) = \langle 0 + A, -e^{0} + B \rangle$, we find that $A = 0$ and $B = 1$.
Hence $\mathbf{v} = \langle 2t, 1 - e^{-t} \rangle$.
We antidifferentiate again to obtain the position vector $\mathbf{s} = \langle t^2 + C, t + e^{-t} + D \rangle$
At $t = 0$, $\mathbf{s} = \langle 3, 3 \rangle$, so $\langle C, 1 + D \rangle = \langle 3, 3 \rangle$ and $C = 3, D = 2$.
$\mathbf{s} = \langle t^2 + 3, t + e^{-t} + 2 \rangle$. When $t = 2$, $\mathbf{s} = \langle 7, 4 + e^{-2} \rangle$

y = s(t)



$0 \le x \le 5$
$-2 \le y \le 3$

Exam III
Section II
Part A — Calculators Permitted

1. p. 69

(a) *Solution I.* Calculate left- and right-hand slopes of the segments joining (1.4, f(1.4)) to the points on either side. Then average the results.

$$\frac{f(1.4)-f(1.2)}{1.4-1.2} = \frac{2.6-3.5}{1.4-1.2} = \frac{-0.9}{0.2} = -4.5$$

$$\frac{f(1.6)-f(1.4)}{1.6-1.4} = \frac{2.0-2.6}{1.6-1.4} = \frac{-0.6}{0.2} = -3$$

The average of these slopes gives $f'(1.4) \approx -3.75$, but either is acceptable.

Solution II. Since the inputs $x = 1.2$ and $x = 1.6$ are symmetrically placed on either side of $x = 1.4$, the result above can be obtained more eeasily simply by calculating one slope.

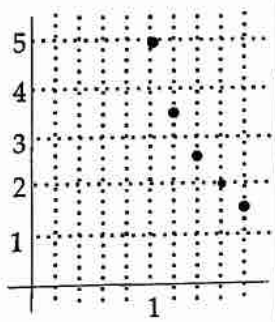
$$\frac{f(1.6)-f(1.2)}{1.6-1.2} = \frac{2.0-3.5}{1.6-1.2} = \frac{-1.5}{0.2} = -3.75$$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(b) Since $f(1.4) = 2.6$ and (from part (a)) $f'(x) \approx -3.75$, we have $y - 2.6 = -3.75(x - 1.4)$.

$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent equation} \end{cases}$

(c) The given points are plotted in the graph to the right. Δx remains constant at 0.2 as we move from $x = 1.0$ to 1.2 to 1.4 to 1.6 to 1.8.
At the same time, Δy increases.
The first Δy is -1.5.
The second Δy is -0.9.
The third Δy is -0.6.
The last Δy is -0.4.
Since $\frac{\Delta y}{\Delta x}$ is increasing, $f''(1.4) > 0$.



$3: \begin{cases} 1: f'(x) \text{ is increasing} \\ 1: \text{answer} \\ 1: \text{explanation} \end{cases}$

(d) With two equal subdivisions, each piece has width $\Delta x = 0.4$.
The midpoints of the two subintervals are $x = 1.2$ and $x = 1.6$.
Then $M_2 = 0.4\,[f(1.2) + f(1.4)] = 0.4\,(3.5 + 2.0) = 2.2$.

$2: \begin{cases} 1: 0.4[f(1.2) + f(1.4)] \\ 1: \text{answer} \end{cases}$

2. p. 70

(a) $\begin{cases} xy = \sqrt{2} \\ x^2 + y^2 = 1 \end{cases}$

$y = \frac{\sqrt{2}}{x}$

$x^2 - \frac{2}{x^2} = 1$

$x^4 - x^2 - 2 = 0$

$(x^2 - 2)(x^2 + 1) = 0$

$x = \pm\sqrt{2}$

The intersections are at $(\sqrt{2}, 1)$ and $(-\sqrt{2}, -1)$.

Alternatively, graph $y_1 = \frac{\sqrt{2}}{x}$ with each of $y_2 = \sqrt{x^2 - 1}$ and $y_3 = -\sqrt{x^2 - 1}$ to obtain intersection points (1.414,1) and (−1.414,−1).

$2: \begin{cases} 1: (\sqrt{2}, 1), \\ 1: (-\sqrt{2}, -1) \end{cases}$

(b) Differentiate the two equations implicitly, and determine $\frac{dy}{dx}$ for each.

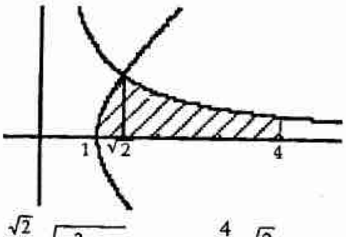
$xy = \sqrt{2} \quad \Rightarrow \quad y + x\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$

$x^2 - y^2 = 1 \quad \Rightarrow \quad 2x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$

Since $\left(-\frac{y}{x}\right)\cdot\left(\frac{x}{y}\right) = -1$, the tangent lines are perpendicular.

$3: \begin{cases} 1: \frac{d}{dx}\left(xy = \sqrt{2}\right) \\ 1: \frac{d}{dx}\left(x^2 - y^2 = 1\right) \\ 1: \text{answer} \end{cases}$

(c)



$\int_1^{\sqrt{2}} \sqrt{x^2 - 1}\, dx + \int_{\sqrt{2}}^{4} \frac{\sqrt{2}}{x}\, dx \approx 0.2664 + 1.4704 \approx 1.737$

$4: \begin{cases} 1: \text{limits} \\ 2: \text{integrands} \\ 1: \text{answer} \end{cases}$

Section II
Part B — No Calculators

3. p. 71

(a) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 5 + 1\cdot(3+2) = 5 + 5 = 10$

1: answer

(b) f' changes from increasing to decreasing or vice versa at $x = -3$, 1 and 3. Thus the graph of f has points of inflection at $x = -3$, 1 and 3.

$2: \begin{cases} 1: x = -3, \ 1 \text{ and } 3 \\ 1: \text{justification} \end{cases}$

(c) f is increasing $\Leftrightarrow f'(x) > 0$. This occurs on intervals $(-6,-4)$ and $(-2, 6)$.
The graph of f is concave down where the graph of f' is decreasing.
This occurs on intervals $(-6,-3)$ and $(1, 3)$.
Thus, graph of f is increasing and concave down on intervals $(-6,-4)$ and $(1, 3)$.

$3: \begin{cases} 1: \text{increasing intervals} \\ 1: \text{concave down} \\ 1: \text{answer} \end{cases}$

(d) $H'(x) = f'[g(x)]\cdot g'(x)$. At $x = 4$, $H'(4) = f'[g(4)]\cdot g'(4)$.
With $g(x) = x^2 - 3x - 1$, $g(4) = 3$ and $g'(4) = 5$, we have
$H'(4) = f'[3]\cdot 5 = 2\cdot 5 = 10$

$3: \begin{cases} 1: H'(x) = f'[g(x)]\cdot g'(x) \\ 1: g(4),\ g'(4) \\ 1: \text{answer} \end{cases}$

4. p. 72

(a) The series diverges.

Solution I. Use the Limit Comparison Test.

We know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, for this is a p-series with $p = \frac{1}{2}$ [p-series diverge if $p < 1$].

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{2n+5}}} = \lim_{n\to\infty} \sqrt{\frac{2n+5}{n}} = \lim_{n\to\infty} \sqrt{2+\frac{5}{n}} = \sqrt{2}$$

Since this limit is finite and positive, the series

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+5}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ exhibit the same behavior.

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+5}}$

Solution II. Use the Direct Comparison Test.

$\sqrt{2n+5} < n$ for $n \geq 4$. Hence $\frac{1}{\sqrt{2n+5}} > \frac{1}{n}$ for $n \geq 4$.

Then the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ implies the divergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+5}}$.

4: { 1: comparsion; 1: compute limit; 1: answer; 1: justification }

(b) We use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{3^n \cdot x^n} \right| = \lim_{n\to\infty} \left| 3x \cdot \frac{n}{n+1} \right|$$
$$= 3|x|$$

There is absolute convergence if $3|x| < 1$; that is, if $-\frac{1}{3} < x < \frac{1}{3}$.

Check endpoints:

$x = \frac{1}{3} \Rightarrow \sum \frac{1}{n}$. This series diverges. It is the harmonic series.

$x = -\frac{1}{3} \Rightarrow \sum (-1)^n \frac{1}{n}$. This converges. It is the alternating harmonic series.

The interval of convergence is $-\frac{1}{3} \leq x < \frac{1}{3}$.

5: { 1: sets up ratio test; 1: computes limit; 1: conclusion of ratio test; 2: endpoint conclusio — 1: right endpoint; 1: left endpoint }

5. p. 73

(a) $a(t) = 6(2t+1)^{-3/2}$

$v(t) = \int 6(2t+1)^{-3/2}\,dt = 3\,\dfrac{(2t+1)^{-1/2}}{-1/2} + C = -\dfrac{6}{\sqrt{2t+1}} + C$

$v(4) = 1 \Rightarrow 1 = -\dfrac{6}{3} + C$

$\Rightarrow 1 = -2 + C$

$\Rightarrow C = 3$

Thus $v(t) = -\dfrac{6}{\sqrt{2t+1}} + 3$

2: { 1: antiderivative; 1: answer }

(b) $v(t) = 0 \Rightarrow -\dfrac{6}{\sqrt{2t+1}} + 3 = 0$

$\Rightarrow 3 = \dfrac{6}{\sqrt{2t+1}} \Rightarrow \sqrt{2t+1} = 2$

$\Rightarrow 2t + 1 = 4 \Rightarrow t = \dfrac{3}{2}$

2: { 1: sets v(t) = 0; 1: answer }

(c) $s(t) = \int\left(-6(2t+1)^{-1/2} + 3\right)dt = -3\,\dfrac{(2t+1)^{1/2}}{1/2} + 3t + D$

$= -6\sqrt{2t+1} + 3t + D$

$s(4) = 9 \Rightarrow 9 = -6\cdot 3 + 12 + D$

$\Rightarrow 9 = -18 + 12 + D$

$\Rightarrow D = 15$

Thus $s(t) = -6\sqrt{2t+1} + 3t + 15$

2: { 1: antiderivative; 1: answer }

(d) *Solution I.* The total distance traveled from $t = 0$ to $t = 4$ is

$$\int_0^4 |v(t)|\,dt = \int_0^4 \left| -\frac{6}{\sqrt{2t+1}} + 3 \right| dt$$

$$= \int_0^{1.5} \left[\frac{6}{\sqrt{2t+1}} - 3\right] dt = \int_{1.5}^{4} \left[-\frac{6}{\sqrt{2t+1}} + 3\right] dt$$

From part (c),

$$= \left[6\sqrt{2t+1} - 3\right]_0^{1.5} + \left[-6\sqrt{2t+1} + 3\right]_{1.5}^{4}$$

$$= 12 - \frac{9}{2} - 6 - 6 + 12 - \frac{9}{2} = 3.$$

3: { 2: distance integral (1: limits, 1: integrand); 1: answer }

Solution II. The particle travels left for $0 < t < 1.5$ and right for $1.5 < t < 4$.
From part (c), $s(0) = 9$; $s(1.5) = 7.5$; $s(4) = 9$.
Hence the distance traveled for $0 \le t \le 1.5$ is $9 - 7.5 = 1.5$;
the distance traveled for $1.5 \le t \le 4$ is $9 - 7.5 = 1.5$.
The total distance traveled is $1.5 + 1.5 = 3$.

6. p. 74

(a) Since $h(x) = \int_0^{x^2} f(t)\,dt$, we have $h(-x) = \int_0^{(-x)^2} f(t)\,dt = \int_0^{x^2} f(t)\,dt = h(x)$.
Since $h(-x) = h(x)$ for every x in the domain, h is an even function.

1: justification

(b) With x positive, h only makes sense if $x^2 \le 5$ (since the domain of the integrand function f is $[0, 5]$.
Hence h is defined for $0 \le x \le \sqrt{5}$.
Since h is even, we then have h defined for all x satisfying $-\sqrt{5} \le x \le \sqrt{5}$.

1: answer

(c) Let the function F be defined by $F(x) = \int_0^{x} f(t)\,dt$.

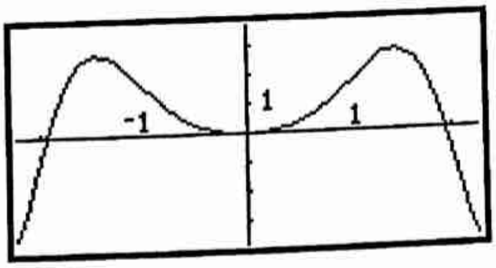
Then $h(x) = F(x^2)$.
By the Chain Rule, $h'(x) = F'(x^2) \cdot 2x$.
But $F'(x) = f(x)$. Hence $h'(x) = f(x^2) \cdot 2x$.
Then $h'(2) = f(4) \cdot 4 = (-3) \cdot 4 = -12$.

2: { 1: chain rule
1: answer }

(d) $h'(x) = f(x^2) \cdot 2x$.
The critical numbers for h are $x = 0, \pm\sqrt{2}$.
h is increasing on $[0, \sqrt{2}]$ and decreasing on $[\sqrt{2}, \sqrt{5}]$. Hence h has its absolute maximum at $x = \sqrt{2}$.
Since h is even, h also has its absolute maximum at $x = -\sqrt{2}$.

4: { 1: sets $h'(x) = 0$
1: critical numbers
1: evaluation at critical pts and endpts
1: answer }

(e)



1: graph

Exam IV
Section I
Part A — No Calculators

1. D p. 75

This is an improper integral.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[-e^{-b}+1\right]$$

$$= \lim_{b\to\infty}\left(-\frac{1}{e^b}+1\right) = 1$$

2. B p. 75

f(x) = x sin x

f'(x) = sin x + x cos x

f"(x) = cos x + cos x – x sin x = 0

2 cos x = x sin x

$$\frac{2}{x} = \frac{\sin x}{\cos x}$$

$$\frac{2}{x} = \tan x$$

3. C p. 76

$$A = \int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_0^4 (2x+1)^{1/2} = \frac{1}{2}\left[(2x+1)^{3/2}\cdot\frac{2}{3}\right]_0^4$$

$$= \frac{1}{3}[27 - 1] = \frac{26}{3}.$$

4. B p. 76

$$\lim_{x\to 3}\frac{\ln\frac{x-1}{2}}{3-x}$$ takes on the indeterminate form $\frac{0}{0}$.

Hence we use L'Hôpital's Rule. First we rewrite the logarithm of the quotient. This leads to:

$$\lim_{x\to 3}\frac{\ln\frac{x-1}{2}}{3-x} = \lim_{x\to 3}\frac{\ln(x-1)-\ln 2}{3-x} = \lim_{x\to 3}\frac{\frac{1}{x-1}}{-1} = -\frac{1}{2}$$

5. D p. 76

The general term of the series is $(-1)^{n+1}\frac{x^n}{n}$ for n ≥ 1

Using the Ratio test: $\left|\frac{x^{n+1}}{n+1}\cdot\frac{x^n}{n}\right| = \frac{n}{n+1}\cdot|x|$, so that $\lim_{n\to\infty}\frac{n}{n+1}\cdot|x| = |x|$.

Thus the series converges if $|x| < 1$ or $-1 < x < 1$. Checking endpoints:

If x = –1, the series $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$ diverges. If x = 1, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is the convergent alternating harmonic series. The interval of convergence for the series is $-1 < x \le 1$

6. C p. 77

I. Extend the direction line at (3,3). It almost goes through the origin. **True**

II. Near the horizontal line $y = 8$, the direction lines near flatten out toward the horizontal. Thus $\frac{dy}{dx} \to 0$. **True**

III. At a given value of x (i.e., along a vertical line) the slopes vary. **False**

---

7. C p. 77

$\lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$ is the <u>definition</u> of the derivative of $\tan x$.

Hence the answer is $\sec^2 x$.

---

8. E p. 77

$$\int \frac{x^2 + 2x + 9}{x^2 + 9}\,dx = \int \left[\frac{x^2 + 9}{x^2 + 9} + \frac{2x}{x^2 + 9} \right] dx$$

$$= \int \left[1 + \frac{2x}{x^2 + 9} \right] dx \quad = x + \ln(x^2 + 9) + C$$

---

9. D p. 78

$y = \ln(\cos x) \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x$

Then $\frac{d^2y}{dx^2} = -\sec^2 x$

---

10. D p. 78

I. Using the slope of the graph, $f'(3) = -1$; $f'(1) = 1$. **False**

II. Using the area under the graph, $\int_0^2 f(x)\,dx = 0$; $f'(3.5) = -1$. **True**

III. $\int_1^0 f(x)\,dx = -\int_0^1 f(x)\,dx = -(-\frac{1}{2}) = \frac{1}{2}$ and $\int_2^3 f(x)\,dx = \frac{1}{2}$ **True**

---

11. C p. 78

The cross-sectional area at coordinate x is given by $A = \left(\sqrt{\sin x}\right)^2 = \sin x$.

Then the volume of the solid is

$$V = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -\cos \pi + \cos 0 = 2$$

---

12. D p. 79

$x(t) = (t-2)(t-6)^3$

$$\begin{aligned} v(t) = x'(t) &= (t-6)^3 + 3(t-6)^2(t-2) \\ &= (t-6)^2[(t-6) + 3(t-2)] \\ &= (t-6)^2[4t - 12] \\ &= 4(t-6)^2(t-3) \end{aligned}$$

The particle moves toward the left if the velocity is negative.
This occurs when $0 \le t < 3$.

---

13. B p. 79

Use Integration by Parts. Take $\left\{ \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x^3 & v = \frac{x^4}{4} \end{array} \right\}$

Then $\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4} dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

14. C p. 79

$y(\cos x) + e^y = 5$ When $x = \frac{\pi}{2}$, $e^y = 5$, so $y = \ln 5$.

Differentiating the given equation,

$\frac{dy}{dx}(\cos x) - y(\sin x) + e^y \cdot \frac{dy}{dx} = 0$

At $(\frac{\pi}{2}, \ln 5)$, this gives $\frac{dy}{dx} \cdot 0 - (\ln 5) \cdot 1 + 5 \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{\ln 5}{5}$.

15. E p. 80

$g'(x) = 0$ means the tangent line is horizontal. $g''(x) < 0$ means the curve is concave down. This happens at point E.

16. B p. 80

Let x denote the x-coordinate of the lower right-hand vertex of the rectangle. Then the area is given by $A(x) = 2xy = 2x \cdot \frac{1}{1 + 2x^2} = \frac{2x}{1 + 2x^2}$.

$A'(x) = \frac{(1 + 2x^2) \cdot 2 - 2x \cdot (4x)}{(1 + 2x^2)^2} = \frac{2 - 4x^2}{(1 + 2x^2)^2}$.

The critical numbers occur when $x^2 = \frac{1}{2}$. When $x < \frac{1}{\sqrt{2}}$ then $A'(x) > 0$ and when $x > \frac{1}{\sqrt{2}}$ then $A'(x) < 0$. When $x^2 = \frac{1}{2}$ the area is a maximum. Height: $y = \frac{1}{1 + 2 \cdot \frac{1}{2}} = \frac{1}{2}$.

17. C p. 80

$f(x) = \frac{x^2 + 10}{e^x} = (x^2 + 1)e^{-x}$;; $f'(x) = 2xe^{-x} - (x^2 + 1)e^{-x} = \frac{2x - x^2 - 1}{e^x} = -\frac{(x - 1)^2}{e^x}$.

Thus, $f'(x) < 0$ for all $x \neq 1$. Hence, f is decreasing everywhere, except when $x = 1$.

$f''(x) = (2 - 2x)e^{-x} + (2x - x^2 - 1)(-e^{-x}) = \frac{2 - 2x - 2x + x^2 + 1}{e^x}$

$= \frac{x^2 - 4x + 3}{e^x} = \frac{(x - 3)(x - 1)}{e^x}$

f is concave down when $f''(x) < 0$; that is, when $1 < x < 3$.

18. B p. 81

Solution I. The point (1,2) is on the graph of f. Hence (2,1) is on the graph of $f^{-1} = h$. $h'(2) = \frac{1}{f'(1)}$. But $f'(x) = 3x^2$, so $f'(1) = 3$. Hence $h'(2) = \frac{1}{3}$.

Solution II. The inverse of f is $h(x) = f^{-1}(x) = (x - 1)^{1/3}$.
Then $h'(x) = \frac{1}{3}(x - 1)^{-2/3}$. Hence $h'(2) = \frac{1}{3} \cdot (1)^{-2/3} = \frac{1}{3}$.

19. D p. 81

Solution I.

$$\int_0^4 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^4 (6-x)\,dx$$
$$= \frac{x^3}{3}\Big|_0^2 + \left(6x - \frac{x^2}{2}\right)\Big|_2^4$$
$$= \frac{8}{3} + (24-8) - (12-2) = 8\frac{2}{3}.$$

Solution II. Counting squares in the graph to the right we see that the value of the definite integral is about $8\frac{2}{3}$

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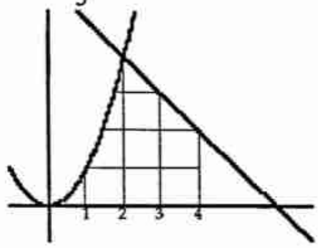


20. E p. 82

$$f(x) = \frac{x}{x-2} = \frac{(x-2)+2}{x-2} = 1 + \frac{2}{x-2}$$

Hence $f'(x) = -\dfrac{2}{(x-2)^2}$.

The slope of the tangent line at $(1,-1)$ is $m = -2$. The slope of the normal is then $m = \frac{1}{2}$.

An equation of the normal is
$$y + 1 = \frac{1}{2}(x-1).$$
$$2y + 2 = x - 1$$
$$0 = x - 2y - 3$$

21. D p. 82

$\mathbf{v}(t) = \langle 1 + t, t^3 \rangle$

$\mathbf{R}(t) = \langle t + \frac{t^2}{2} + C, \frac{t^4}{4} + D \rangle$

When $t = 0$, $\mathbf{R}(t) = \langle C, D \rangle$.
Since we are given that $\mathbf{R}(0) = \langle 5, 0 \rangle$, we know $C = 5$ and $D = 0$.

Hence $\mathbf{R}(t) = \langle t + \frac{t^2}{2} + 5, \frac{t^4}{4} \rangle$. From this we obtain $\mathbf{R}(2) = \langle 9, 4 \rangle$.

22. E p. 82

$f(x) = x^{2/3}(2x-5) = 2x^{5/3} - 5x^{2/3}$

Then $f'(x) = \frac{10}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{10}{3}x^{-1/3}(x-1)$

$f'(x)$ is positive if $x > 1$ or $x < 0$.
$f'(x)$ is negative if $0 < x < 1$. This is when f is decreasing.

23. A p. 83

$$y_{ave} = \frac{1}{1/2 - 0}\int_0^{1/2}(e^{2x}+1)\,dx = 2\left[\frac{1}{2}e^{2x} + x\right]_0^{1/2} = 2[(\tfrac{1}{2}e + \tfrac{1}{2}) - (\tfrac{1}{2} + 0)]$$
$$= e + 1 - 1 = e$$

24. B p. 83

f(2) must equal $\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x-2}$ Rationalize the numerator.

$$\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x-2} = \lim_{x\to 2} \frac{(x+2)-2x}{(x-2)\left[\sqrt{x+2}+\sqrt{2x}\right]}$$

$$= \lim_{x\to 2} \frac{2-x}{(x-2)\left[\sqrt{x+2}+\sqrt{2x}\right]}$$

$$= \lim_{x\to 2} \frac{-1}{\left[\sqrt{x+2}+\sqrt{2x}\right]} = -\frac{1}{4}$$

---

25. B p. 83

$y = \sqrt{x^2+3}$

$\frac{dy}{dx} = \frac{2x}{2\sqrt{x^2+3}} = \frac{x}{\sqrt{x^2+3}}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2}$

The tangent to the curve at (1,2) is $y-2 = \frac{1}{2}(x-1)$.

When $x = 1.04$, $y = 2 + \frac{1}{2}(0.04) = 2.02$.

---

26. D p. 84

$y(0) = 2$ and $y'(0) = 0 + 2 = 2$

Then $y(.5) \approx 2 + 2 \cdot (.5) = 3$

$y(.5) = 3$ and $y'(.5) = .5 + 3 = 3.5$

Then $y(1) \approx 3 + 3.5 \cdot (.5) = 4.75 = \frac{19}{4}$.

---

27. D p. 84

I.	Direct Comparsion Test	**True**
II.	$a = \frac{4}{3}, r = \frac{1}{3}, \frac{a}{1-r} = \frac{4/3}{1-1/3} = 2$	**True**
III.	Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\lim_{n\to\infty} \frac{1}{n} = 0$	**False**

---

28. B p. 84

The distance a car travels in a given time is the area under its velocity function.

I.	Car 1's velocity is greater than Car 2's velocity over the interval.	**True**
II.	On the interval [0,2], the area under Car 2's velocity function is greater than the area under Car 1's velocity function.	**True**
III.	Acceleration is the derivative of velocity. The cars' velocity functions have different slopes at time $t = 1$.	**False**

Exam IV
Section I
Part B — Calculators Permitted

1. D p. 85

The distance traveled is $\int_0^{10} v(t)\,dt$.

We use the Trapezoid Rule, with $n = 5$, to approximate this.

$$T_5 = \frac{h}{2}[\,v(0) + 2\cdot v(2) + 2\cdot v(4) + 2\cdot v(6) + 2\cdot v(8) + v(10)\,]$$

$$= \frac{2}{2}[\,30 + 2(36) + 2(40) + 2(48) + 2(54) + 60\,] = 446$$

---

2. D p. 85

$$f(x) = \frac{\ln x^2 - x\ln x}{x-2}, \qquad x \neq 2$$

$$= \frac{2\ln x - x\ln x}{x-2}$$

$$= \frac{(2-x)\ln x}{x-2}$$

$$= -\ln x$$

To have continuity at $x = 2$, $\lim_{x\to 2} f(x) = f(2)$.

Since $\lim_{x\to 2} f(x) = -\ln 2$, we must have $f(2) = -\ln 2$.

---

3. C p. 86

I. f'' does not change sign at $x = -2$. Hence f has the same concavity on each side of $x = -2$. **False**

II. $f''(x) > 0$ on the interval $(0,3)$, so the graph of f is concave up there. **False**

III. $f''(x) > 0$ on $(0,3)$, so f' is increasing from 0 to 3. Since $f'(0) = 0$, $f'(2)$ must be positive. Then f is increasing at 2. **True**

---

4. C p. 86

On the interval $0 \le t \le 21$ the graph of $R(t) = 3\cos\left(\frac{t}{3}\right)$ has zeros at $t = 4.71$ and $t = 14.14$.

$R(t)$ is positive on $0 < t < 4.71$, negative on $4.71 < t < 14.14$ and positive on $14.14 < t < 21$. Thus the number of flies increases, then decreases, then increases again.

Let $N(t)$ = the number of flies at time t, then $N(t) = N(0) + \int_0^t 3\sqrt{x}\cos\left(\frac{x}{3}\right)dx$

At $t = 0$, $N(0) = 500$ flies.
At $t = 4.71$, $N(4.71) = 500 + 10.97 \approx 511$ flies.
At $t = 14.14$, $N(14.14) = 500 - 143.92 \approx 456$ flies.
At $t = 21$, $N(21) = 500 + 19.65 \approx 520$ flies. This is the maximum number of flies.

---

5. C p. 86

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ In the given formula, let $a = x$ and $b = h$.

$$= \lim_{h\to 0} \frac{3x^2h + 2h^2}{h} = \lim_{h\to 0} (3x^2 + 2h) = 3x^2$$

6. A p. 87

The velocity vector is given by $\begin{cases} \dfrac{dx}{dt} = 2t + 2 \\ \dfrac{dy}{dt} = 4t - 6 \end{cases}$

When $t = 2$, the velocity vector is $\mathbf{v}(2) = \langle 6, 2 \rangle$.

The speed of the particle is the length of this vector, $|\mathbf{v}| = \sqrt{6^2 + 2^2} = 2\sqrt{10}$.

7. C p. 87

$y = x^2 + 1$

Solution I. Since the distance of a point (x,y) from the origin is $D = \sqrt{x^2 + y^2}$, we have in this case

$$D = \sqrt{x^2 + (x^2+1)^2} = \sqrt{x^4 + 3x^2 + 1}.$$

Then $$\frac{dD}{dt} = \frac{(4x^3 + 6x)\frac{dx}{dt}}{2\sqrt{x^4 + 3x^2 + 1}}.$$

When $x = 1$ and $\frac{dx}{dt} = \frac{3}{2}$, this gives $$\frac{dD}{dt} = \frac{10 \cdot \frac{3}{2}}{2\sqrt{5}} = \frac{3}{2}\sqrt{5} \approx 3.354.$$

Solution II. Since $y = x^2 + 1$, we have $\frac{dy}{dt} = 2x\frac{dx}{dt}$.

Again, the distance of a point (x,y) from the origin is $D = \sqrt{x^2 + y^2}$. This time we differentiate implicitly to get

$$\frac{dD}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}} = \frac{x\frac{dx}{dt} + y\left(2x\frac{dx}{dt}\right)}{\sqrt{x^2+y^2}} = \frac{x + 2xy}{\sqrt{x^2+y^2}}\frac{dx}{dt}.$$

Since $\frac{dx}{dt} = \frac{3}{2}$, then at the point $(1, 2)$ we have

$$\frac{dD}{dt} = \frac{1+4}{\sqrt{5}} \cdot \frac{3}{2} = \frac{3}{2}\sqrt{5} \approx 3.354.$$

8. E p. 87

$$F(x) = \int_0^x \frac{\sin t}{1+\cos t}\,dt \quad \Rightarrow \quad F'(x) = \frac{\sin x}{1+\cos x}$$

Solution I. Enter Y1 = F'(x) and evaluate nDerive(Y1,x,$\frac{\pi}{3}$) = 0.667

Solution II. $F''(x) = \dfrac{(1+\cos x)\cos x - \sin x\,(-\sin x)}{(1+\cos x)^2} = \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

$$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

$$F''\left(\frac{\pi}{3}\right) = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$

---

9. B p. 88

$u = \sqrt{x+1}$

$u^2 - 1 = x$

$2u\,du = dx$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 2$

$$\int_1^2 \frac{1}{(u^2-1)\cdot u} \cdot 2u\,du = \int_1^2 \frac{2}{u^2-1}\,du$$

---

10. C p. 88

The average values of the functions f and g on [0,b] are

$$\frac{1}{b}\int_0^b \cos(2x)\,dx \text{ and } \frac{1}{b}\int_0^b (e^x - 1)\,dx .$$

Setting these equal, we have $\dfrac{1}{b}\displaystyle\int_0^b \cos(2x)\,dx = \dfrac{1}{b}\int_0^b (e^x-1)\,dx$

$$\int_0^b \cos(2x)\,dx = \int_0^b (e^x - 1)\,dx$$

$$\frac{1}{2}\sin(2x)\Big|_0^b = (e^x - x\Big|_0^b$$

$$\frac{1}{2}\sin(2b) = (e^b - b) - (1 - 0)$$

$$\frac{1}{2}\sin(2b) = e^b - b - 1$$

The intersection of the graphs of $y_1 = \frac{1}{2}\sin(2x)$ and $y_2 = e^x - x - 1$ occurs at $x \approx 0.854$.

Alternatively, the zero of $y = \frac{1}{2}\sin(2x) - e^x - x - 1$ occurs at $x \approx 0.854$.

11. D p. 89

The series converges absolutely if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1.$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+2)\,3^{n+1}} \cdot \frac{(n+1)\,3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \cdot \frac{n+1}{n+2} \right| = \frac{|x|}{3}$$

$$\frac{|x|}{3} < 1 \quad \Leftrightarrow \quad |x| < 3$$

This describes an interval centered at the origin and having a radius of 3.
Thus the radius of convergence of the series is R = 3.

---

12. D p. 89

I. $$y' = \frac{2x}{\sqrt{1-(x^2-1)^2}} - 1 = \frac{2x}{\sqrt{1-x^4+2x^2-1}} - 1 = \frac{2x}{\sqrt{2x^2-x^4}} - 1 = \frac{2x}{|x|\sqrt{2-x^2}} - 1$$

$\lim_{x\to 0^+} y' = \frac{2}{\sqrt{2}} - 1$ because when $x > 0$, $\frac{x}{|x|} = 1$.

$\lim_{x\to 0^-} y' = -\frac{2}{\sqrt{2}} - 1$ because when $x < 0$, $\frac{x}{|x|} = -1$.

Since these limits are different, the derivative at $x = 0$ does not exist. No

II. $$y = \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad y' = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Since $\lim_{x\to 0^+} y' = 0 = \lim_{x\to 0^-} y'$, this does have a derivative at $x = 0$. Yes

III. $y = \sqrt{x^4} = x^2$. This is differentiable everywhere. Yes

---

13. C p. 89

With $f(x) = \frac{4\ln x}{e^{\sqrt{x}}}$, we have

$$F(3) - F(1) = \int_1^3 f(x)\,dx$$

$$F(3) = F(1) + \int_1^3 f(x)\,dx = 2 + 1.177 = 3.177$$

---

14. C p. 90

$$T_4 = \frac{1}{2}[\,0 + 2\,(1.1) + 2\,(1.4) + 2\,(1.2) + 1.5\,]$$
$$= \frac{1}{2}[\,2.2 + 2.8 + 2.4 + 1.5\,]$$
$$= \frac{1}{2}[\,8.9\,] = 4.45$$

---

15. B p. 90

$f'(x) = x\,(x-a)^3$ with $a > 0$.

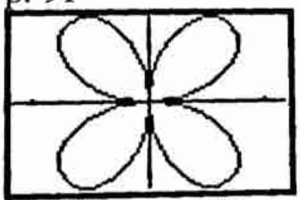
I. $f'(2a) = 2a\,(a^3) = 2a^4 > 0$ — True

II. $f'(0) = 0$.
$f'(x) > 0$ if $x < 0$. f is increasing to the left of $x = 0$.
$f'(x) < 0$ if $0 < x < a$. f is decreasing to the right of $x = 0$. — True

III. $f''(x) = (x-a)^3 + x \cdot 3\,(x-a)^2$
$= (x-a)^2\,[(x-a) + 3x]$
$= (x-a)^2\,(4x-a)$
$f''(a) = 0$, but $f''(x)$ does not change sign at $x = a$. — False

---

16. D p. 91



$-1.175 \le x \le 1.175$
$-.775 \le y \le .775$

The curve is a four-leaved rose.
The area is determined by evaluating

$$\int_0^{2\pi} \frac{1}{2}[r(\theta)]^2\,d\theta = \int_0^{2\pi} \frac{1}{2}[\sin(2\theta)]^2\,d\theta.$$

With a calculator, we determine the value of this definite integral to be approximately 1.571.

---

17. D p. 91

The average rate of change of f on the interval $[3,x]$ is defined to be $\frac{f(x)-f(3)}{x-3}$.

In this problem, we have $\frac{f(x)-f(3)}{x-3} = \frac{x^2-x-6}{x-3}$.

Also by an alternate definition of the derivative, $f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

Hence $f'(3) = \lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3}(x+2) = 5$.

Exam IV
Section II
Part A — Calculators Permitted

1. p. 93

(a) Graph $f(x) = \frac{1}{2}e^{x/2} - \ln(x^3 + 1)$ and

$f'(x) = \frac{1}{2}e^{x/2} - \frac{3x^2}{x^3 + 1}$ with the window

$[-1, 5] \times [-2, 5]$. The zeros of f' are $x = -0.364$, $x = 0.487$ and $x = 1.977$. The relative minimums are at $x = -0.364$ and $x = 1.977$ because $f'(x)$ shows changes in sign from minus to zero to plus at each of these values of x. The relative maximum is at $x = 0.487$ because the sign of $f'(x)$ changes from plus to zero to minus

y = f'(x)

$[-1, 5] \times [-2, 5]$

$3: \begin{cases} 1: \text{solves } f'(x) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(b) The graph of f is increasing when $f'(x) > 0$.

This happens when $-0.364 < x < 0.487$ and $x > 1.977$.

$2: \begin{cases} 1: f'(x) > 0 \\ 1: \text{answer} \end{cases}$

(c) $f''(x) = \frac{1}{4}e^{x/2} - \frac{(x^3+1)(6x) - 3x^2(3x^2)}{(x^3 + 1)^2} = \frac{1}{4}e^{x/2} - \frac{(6x^4+6x) - 9x^4}{(x^3 + 1)^2}$

$= \frac{1}{4}e^{x/2} - \frac{6x - 3x^4}{(x^3 + 1)^2}$. The graph of f is concave down when

$f''(x) < 0$. That occurs when $6x - 3x^4 < 0$; that is: $0.0425 < x < 1.0919$.

$2: \begin{cases} 1: f''(x) < 0 \\ 1: \text{answer} \end{cases}$

(d) $\int_0^1 \left[e^{x/2} - \ln(x^3 + 1)\right] dx = 1.097$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

2. p. 94

$$\begin{cases} x = 2 - 3\cos t \\ y = 3 + 2\sin t \end{cases} \quad \text{for } -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

(a) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{3\sin t} = \frac{2}{3}\cot t$

$3: \begin{cases} 1: x'(t) \\ 1: y'(t) \\ 1: \frac{dy}{dx} \text{ in terms of } t \end{cases}$

(b) When $t = \frac{\pi}{4}$, we have the point $\begin{cases} x = 2 - 3\frac{\sqrt{2}}{2} = 2 - \frac{3}{2}\sqrt{2} \\ y = 3 + 2\frac{\sqrt{2}}{2} = 3 - \sqrt{2} \end{cases}$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{2}{3}$ (from part (a)). Hence the tangent line has the equation:

$y - (3 + \sqrt{2}) = \frac{2}{3}(x - 2 + \frac{3}{2}\sqrt{2})$. Multiplying both sides by 3 and collecting terms leads to: $2x - 3y + 5 + 6\sqrt{2} = 0$.

$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent line} \end{cases}$

(c) The curve intersects the y-axis when $x = 0$. That is, $2 - 3\cos t = 0 \Rightarrow]\cos t = \frac{2}{3}$

The two values of t that yield the y-intercepts are $t = \pm \text{Arccos}\frac{2}{3} \approx \pm 0.841069$.

(It's better to use $\pm \text{Arccos}\frac{2}{3}$ and not round off too soon. We did it to save space.)

Arc Length $= \int_{-.841}^{.841} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-.841}^{.841} \sqrt{(3\sin t)^2 + (2\cos t)^2}\, dt \approx 3.757.$

$4: \begin{cases} 2: \text{intersection pts} \\ 1: \text{limits and integrand} \\ 1: \text{answer} \end{cases}$

Exam IV
Section II
Part B — No Calculators

3. p. 95

(a) $\frac{dv}{dt} = \frac{v(30) - v(20)}{30 - 20} = \frac{11 - 7}{10} = 0.4$ ft / sec^2

2: { 1: a difference quotient using numbers from table and interval contains 25; 1: 0.4 ft / sec^2 }

(b) Midpoint sum = $20[v(10) + v(3) + v(50)] = 20(14 + 11 + 40) = 1300$ feet

2: { 1: uses $v(10)$, $v(30)$, $v(50)$; 1: answer }

(c) $v_B(t) = \int \frac{1}{\sqrt{t+9}}\,dt = 2\sqrt{t+9} + C$

$3 = v(0) \Rightarrow 6 + C \Rightarrow C = -3$

$v_B(t) = 2\sqrt{t+9} - 3$

$v_B(40) = 14 - 3 = 11 < 12 = v_A(40)$

Car A is traveling faster at time $t = 40$ seconds,

5: { 1: $2\sqrt{t+9}$; 1: constant; 1: use initial conditions; 1: finds $v_B(40)$; 1: compares to $v_A(40)$ and draws conclusion }

4. p. 96

$$f(x) = \sum_{n=0}^{\infty} a_n x^{2n} \qquad \text{with } a_n = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{a_{n-1}}{n} & \text{if } n \geq 1 \end{cases}$$

(a) $a_0 = 1$

$a_1 = \frac{1}{1} = 1$

$a_2 = \frac{1}{2} = \frac{1}{2!}$

$a_3 = \frac{1/2}{3} = \frac{1}{6} = \frac{1}{3!}$

$a_4 = \frac{1/3!}{4} = \frac{1}{4!}$

$$f(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots + \frac{x^{2n}}{n!} + \ldots$$

3: { 2: first 5 terms; 1: general term }

(b) *Solution I.* We use the ratio test.

We must have $\lim_{n\to\infty} \left| \frac{(n+1)^{st} \text{ term}}{n^{th} \text{ term}} \right| < 1.$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2n+2}}{(n+1)!}}{\frac{x^{2n}}{n!}} \right| \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| = 0$$

Since the limit is 0 for every real x, the series converges for all real x and the radius is infinite.

3: { 1: sets up ratio; 1: limit; 1: answer }

Solution II. Rewrite f(x) as $1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots + \frac{(x^2)^n}{n!} + \ldots$

This is the Maclaurin series for $g(x) = e^{x^2}$. This series is well known to converge for all real x; that is, the radius is infinite.

(c) *Solution I.*

$$f'(x) = 2x + \frac{4x^3}{2!} + \frac{6x^5}{3!} + \frac{8x^7}{4!} + \ldots + \frac{(2n)x^{2n-1}}{n!} + \ldots$$

$$= 2x\left[1 + \frac{2x^2}{2!} + \frac{3x^4}{3!} + \frac{4x^6}{4!} + \ldots + \frac{(n)x^{2n-2}}{n!} + \ldots\right]$$

$$= 2x\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots + \frac{x^{2n}}{n!} + \ldots\right]$$

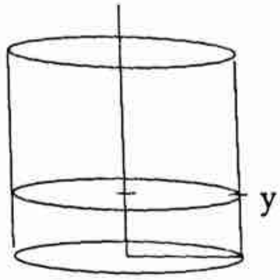
$$= (2x) \cdot f(x)$$

3: { 2: $f'(x)$; 1: conclusion }

Solution II. Since $f(x) = e^{x^2}$, we have $f'(x) = e^{x^2}(2x) = (2x) \cdot f(x)$.

5. p. 97

(a) $\frac{dy}{dt} = -k\sqrt{y}$



Separating variables gives:

$$\frac{dy}{\sqrt{y}} = -k\,dt$$

$$2\sqrt{y} = -kt + C$$

$$\sqrt{y} = -\frac{k}{2}t + \frac{C}{2}$$

$$\sqrt{y} = -\frac{k}{2}t + D$$

$$y = \left(-\frac{k}{2}t + D\right)^2$$

$3:\begin{cases} 1:\text{separates variables} \\ 1:\text{antiderivatives} \\ 1:\text{solves for y} \end{cases}$

(b) $y(0) = 9 \quad \Rightarrow \quad 9 = (D)^2 \quad \Rightarrow \quad D = 3$

$y(20) = 4 \quad \Rightarrow \quad 4 = \left(-\frac{k}{2}\cdot 20 + 3\right)^2$

$\Rightarrow \quad 4 = (-10k + 3)^2$

$\Rightarrow \quad 2 = -10k + 3$

$\Rightarrow \quad \frac{1}{10} = k$

Thus $y = \left(-\frac{1}{20}t + 3\right)^2$

$3:\begin{cases} 1:\text{uses initial condition } y(0) = 9 \\ 1:\text{uses initial condition } y(20) = 4 \\ 1:\text{equation} \end{cases}$

(c) Using the result of part (b), $\frac{dy}{dt} = 2\left(-\frac{t}{20} + 3\right)\left(-\frac{1}{20}\right)$

$$-0.1 = -0.1\left(3 - \frac{t}{20}\right)$$

$$1 = 3 - \frac{t}{20}$$

$t = 40$ minutes after the valve is opened.

$3:\begin{cases} 2:\text{finds } \frac{dy}{dx} \\ 1:\text{answer} \end{cases}$

or

Alternatively one could start with the original differential equation:

$$\frac{dy}{dt} = -k\sqrt{y}.$$

With $-.1$ for $\frac{dy}{dt}$, we have $-.1 = -\frac{1}{10}\sqrt{y}$. Hence $\sqrt{y} = 1$, so $y = 1$.

Returning to the specific solution of the differential equation found in part (b), we have

$$1 = \left(-\frac{1}{20}t + 3\right)^2$$

$$1 = -\frac{1}{20}t + 3$$

$\frac{1}{20}t = 2$, and $t = 40$ minutes.

$3:\begin{cases} 2:\text{solves for y} \\ 1:\text{answer} \end{cases}$

6. p. 98

(a) Calculating the areas of the regions under the graph of f we obtain:

$$G(3) = \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt = \frac{3}{2} + 0 = \frac{3}{2}$$

1: $G(3)$

(b) Using the Fundamental Theorem, $G'(x) = f(x)$.

G is increasing when $G' = f$ is positive-valued.
That occurs on the intervals $(0, \frac{5}{2})$ and $(\frac{19}{4}, 5)$.

G is concave down when $G' = f$ is decreasing.
That occurs on the interval $(2, 4)$.

G has both properties on the interval $(2, \frac{5}{2})$.

4: { 1: $G'(x) = f(x) > 0$; 1: intervals; 1: f decreasing; 1: interval }

(c) $G(3) = \frac{3}{2}$

$G(4) = \frac{3}{2} - 2 = -\frac{1}{2}$

G is continuous, since $G' = f$ exists.
The Intermediate Value Theorem guarantees a zero between $x = 3$ and $x = 4$.

G is monotone decreasing on the interval $(3, 4)$ since $G' = f$ is negative-valued there. Hence G can only have the value 0 once in that interval.

1: analysis

(d) $G(3) = \frac{3}{2}$

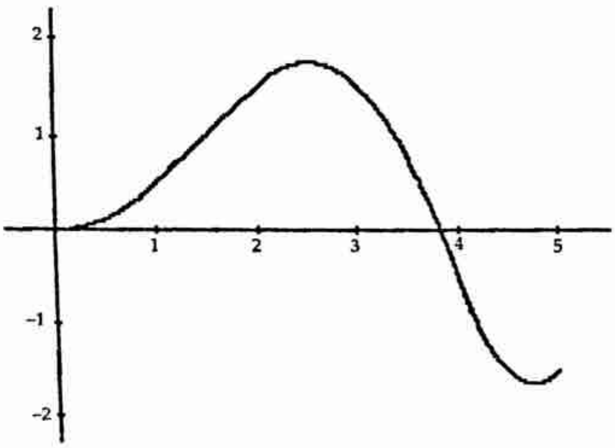
$G'(3) = f(3) = -1$

An equation of the tangent to the graph of G at $(3, \frac{3}{2})$ is

$$y - \frac{3}{2} = -1(x - 3) \text{ or } y = -x + \frac{9}{2}$$

2: { 1: slope; 1: tangent equation }

(e)



1: graph

Exam V
Section I
Part A — No Calculators

1. C p. 99

Rewrite the integrand by dividing.

$$\int_2^3 \frac{x}{x-1}\,dx = \int_2^3 \left(1 + \frac{1}{x-1}\right) dx$$

$$= [x + \ln|x-1|\,]_2^3 = (3 + \ln 2) - (2 + \ln 1) = 1 + \ln 2$$

2. C p. 99

The radius of convergence for the differentied series is the same as the radius of the convergence for original series. Thus, given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ with radius of convergence 1, the differentiated series $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ converges for all x, $-1 < x < 1$.

Checking endpoints: $x = -1 \Rightarrow$ **convergent** alternating harmonic series $\sum_1^{\infty} \frac{(-1)^{n-1}}{n}$

$x = 1 \Rightarrow$ **divergent** harmonic series $\sum_1^{\infty} \frac{1}{n}$

Interval of convergence : $-1 \le x < 1$.

3. C p. 100

The velocity vector is $\mathbf{v}(t) = (6t - 4)\mathbf{i} + (2t + 2)\mathbf{j}$.
At time $t = 2$, the velocity vector is $\mathbf{v}(2) = 8\mathbf{i} + 6\mathbf{j} = \langle 8,6\rangle$.
The speed of the particle is the magnitude (length) of this velocity vector.

Speed = $\sqrt{8^2 + 6^2} = 10$

4. C p. 100

$f(x) = \ln(x^2 - e^{2x}) \quad \Rightarrow \quad f'(x) = \frac{1}{x^2 - e^{2x}} \cdot (2x - 2e^{2x})$

Then $f'(1) = \frac{1}{1 - e^2} \cdot (2 - 2e^2) = \frac{2(1 - e^2)}{1 - e^2} = 2$

5. B p. 100

With $y = \int_0^x \sqrt{\frac{u}{3}}\, du$, we have $\frac{dy}{dx} = \sqrt{\frac{x}{3}}$.

Since arc length is computed as $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, we have

$$L = \int_0^9 \sqrt{1 + \frac{x}{3}}\ dx = 3 \cdot \frac{2}{3} \cdot (1 + \frac{x}{3})^{3/2}\Big|_0^9 = 2(4^{3/2} - 1^{3/2}) = 14$$

---

6. C p. 101

$$\frac{dN}{dt} = 0.05N - 0.0005N2 \Rightarrow \frac{dN}{dt} = 0.05N(1 - \frac{1}{100}N)$$

This is a logistic equation; the carrying capacity is N = 100.

---

7. D p. 101

$v(t) = s'(t) = t^3 - 2t^2 + t$ and $v'(t) = 3t^2 - 4t + 1 = (3t-1)(t-1)$. The critical numbers are $t = \frac{1}{3}$ and $t = 1$. We evaluate at the critical numbers and endpoints and take the largest resulting value. $v(\frac{1}{3}) = \frac{4}{27}$, $v(1) = 0$, $v(0) = 0$, $v(3) = 12$. The maximum velocity is 12 m/s at the right end point.

---

8. E p. 101

The total area of the region between the graph of f and the x-axis is $\int_{-4}^{4} |f(x)|\, dx$.

We evaluate this by counting areas of squares and triangles.

Interval	[−4,−2]	[−2,2]	[2,4]
Area	2	4	3

The total area is 9.

---

9. E p. 102

Differentiate implicitly with respect to t.

$$y^2 + x^2 + y = 10 \quad \Rightarrow \quad 2y\frac{dy}{dt} + 2x\frac{dx}{dt} + \frac{dy}{dt} = 0$$

Using the given values, $x = 2$, $y = 2$, $\frac{dx}{dt} = -5$, we have

$4\frac{dy}{dt} - 20 + \frac{dy}{dt} = 0$. Hence $5\frac{dy}{dt} = 20$, so $\frac{dy}{dt} = 4$.

---

10. C p. 102

If $u = \ln x$, then $x = e^u$ and $dx = e^u\, du$.
In addition, when $x = 1$ then $u = 0$ and when $x = e$ then $u = 1$.
Hence the limits of integration change from [1,e] to [0,1].

Then $\int_1^e \frac{1-\ln x}{x^2}\, dx = \int_0^1 \frac{1-u}{e^{2u}} \cdot e^u\, du = \int_0^1 \frac{1-u}{e^u}\, du$

---

11. D p. 102

Let N denote the number of bacteria present at time t.

Then $\frac{dN}{dt} = 3N + 2.$

Separating variables, we have $\frac{dN}{3N + 2} = dt.$

Integrating, we obtain $\frac{1}{3}\ln|3N + 2| = t + C.$

Then $\ln|3N + 2| = 3t + 3C.$

When $t = 0$, $N = 10$. Hence $\ln 32 = 3C.$

Then $\ln|3N + 2| = 3t + \ln 32.$

At a time t to be determined, there are 42 bacteria present.

Thus $\ln(3 \cdot 42 + 2) = 3t + \ln 32.$

Then $\ln 128 - \ln 32 = 3t,$

so $t = \frac{1}{3}\ln 4.$

---

12. A p. 103

$\frac{dy}{dx} = x \sec y \quad \Rightarrow \quad \cos y\, dy = x\, dx \quad \Rightarrow \quad \sin y = \frac{x^2}{2} + C.$

$y = 0$ when $x = \sqrt{2}$ gives $0 = 1 + C$, so $C = -1$.

Hence $\sin y = \frac{x^2}{2} - 1.$

When $x = 1$, $\sin y = -\frac{1}{2}$, so $y = -\frac{\pi}{6}$.

---

13. C p. 103

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{\sqrt{x}} = 1 + \sqrt{x} + \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^4}{4!}$$ and at $x = 4$, we have

$$e^{\sqrt{4}} = 1 + 2 + \frac{(\sqrt{4})^2}{2!} + \frac{(\sqrt{4})^3}{3!} + \frac{(\sqrt{4})^4}{4!} = 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} = 7$$

---

14. E p. 103

$F(x) = 3x \sin x - \cos x - \frac{3\pi}{2}$

$F'(x) = 3\sin x + 3x \cos x + \sin x = 4\sin x + 3x \cos x$

Then $F'(\frac{\pi}{2}) = 4$. Since $F(\frac{\pi}{2}) = 3\frac{\pi}{2} - \frac{3\pi}{2} = 0$, we need an equation of the line through the point $(\frac{\pi}{2}, 0)$ with slope $m = 4$.

Hence we have $y - 0 = 4(x - \frac{\pi}{2})$, or $y = 4x - 2\pi$.

---

15. D p. 104

To have continuity, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 3.$

Hence $\lim_{x\to 1^-} (4 - x^2) = \lim_{x\to 1^+} (mx + b)$, so $3 = m + b$.

For differentiability, $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$.

Hence , $\lim_{x\to 1^-} -2x = \lim_{x\to 1^+} m$, so $-2 = m$.

Then we have two equations in two unknowns: $\begin{cases} m + b = 3 \\ m = -2 \end{cases}$ The solution is $m = -2,\ b = 5$.

---

16. B p. 104

We break the integrand into partial fractions.

$$\frac{8}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

$8 = A(x + 3) + B(x - 1)$.

When $x = 1$, we have $8 = 4A$, so $A = 2$.

When $x = -3$, we have $8 = -4B$, so $B = -2$.

Then $\int \frac{8}{(x-1)(x+3)}\,dx = \int \left[\frac{2}{x-1} - \frac{2}{x+3}\right] dx$

$= 2\ln|x - 1| - 2\ln|x + 3| + C = 2\ln\left|\frac{x-1}{x+3}\right| + C.$

---

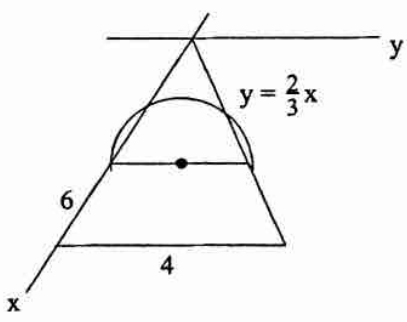
17. A p. 104

$f(x) = \frac{x-k}{x+k} \Rightarrow f'(x) = \frac{(x+k)\cdot 1 - (x-k)\cdot 1}{(x+k)^2} = \frac{2k}{(x+k)^2} = 2k(x+k)^{-2}$

Then $f''(x) = -4k(x+k)^{-3}$. Hence $f''(0) = -4k \cdot k^{-3} = -\frac{4}{k^2}$.

---

18. B p. 105



Locate the triangle in a coordinate system as shown. At any given x-coordinate, the y-coordinate is then $\frac{2}{3}x$, and the radius of the semicircular cross section is $r = \frac{1}{3}x$.

Then the volume of the solid is

$$V = \int_0^6 \frac{1}{2}\pi\left(\frac{1}{3}x\right)^2 dx$$

$$= \frac{\pi}{18} \cdot \frac{x^3}{3}\Big|_0^6 = \frac{\pi}{18}(72 - 0) = 4\pi.$$

---

19. B p. 105

$\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = f'(1)$ if $f(x) = x^3$.

Then $f'(x) = 3x^2$, and $f'(1) = 3$.

20. B p. 106

We need to have both $f'(x) \ge 0$ and $f''(x) > 0$.

$f'(x) = x^2(1 - x) \ge 0$ when $x \le 1$.

$f''(x) = x(2 - 3x) > 0$ when $0 < x < \frac{2}{3}$.

Both properties hold when $0 < x < \frac{2}{3}$.

---

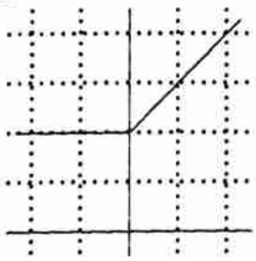
21. D p. 106

Average value $= \frac{1}{b - a}\int_a^b f(x)\,dx$

Hence we compute $\frac{1}{8 - 0}\int_0^8 x^{2/3}\,dx \quad = \frac{1}{8}\cdot\frac{3}{5}x^{5/3}\Big]_0^8 \quad = \frac{3}{40}(32 - 0) = \frac{12}{5}$

---

22. E p. 106



The function f is graphed to the left.

I. F(1) is the area under the graph on the interval [–2,1]. Counting areas of squares and a triangle we see $F(1) = 6.5$. **True**

$F'(1) = f(1) = 3$. **True**

III. $F''(1) = f'(1) =$ the slope of f at $x = 1$. **True**

---

23. B p. 107

I. $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[\frac{x^{-2}}{-2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}$ Convergent

II. $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}[2x^{1/2}]_a^1 = \lim_{a\to 0^+}(2 - 2\sqrt{a}) = 2$ Convergent

III. $\int_0^1 \frac{1}{x^3}\,dx = \lim_{a\to 0^+}\left[\frac{x^{-2}}{-2}\right]_a^1 = \lim_{a\to 0^+}\left(-\frac{1}{2} + \frac{1}{2a^2}\right) = \infty$ Divergent

---

24. D p. 107

$a(t) = \frac{1}{t^2} \Rightarrow v(t) = -\frac{1}{t} + C$

$v(1) = -1 \Rightarrow -1 = -1 + C \Rightarrow C = 0 \Rightarrow v(t) = -\frac{1}{t}$

Then $x(t) = -\ln|t| + D$.

Since $x(1) = 2$, we have $2 = -\ln(1) + D$, so $D = 2$.

Then $x(e) = -\ln(e) + 2 = -1 + 2 = 1$.

---

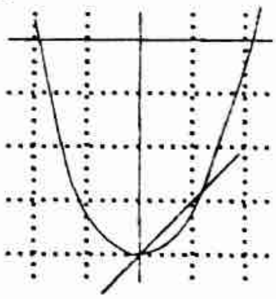
25. B p. 107

From the slope field, we see that $\frac{dy}{dx}$ is periodic and sinusoidal.

The only possible answer among the options is $\frac{dy}{dx} = \sin x$.

---

26. A p. 108



A rough sketch of the region is shown to the left.
The linear function is on top and the quadratic function below on the interval [0,1], which is where a region is enclosed.
The area of the region is

$$\int_0^1 [(x-4)-(x^2-4)]\,dx = \int_0^1 (x-x^2)\,dx$$

---

27. D p. 108

Solution I. From the well known expansion of the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

we have $e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots.$

Then the coefficient of x^3 is $\frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$.

Solution II. If $f(x) = e^{2x}$, then the coefficient of x^3 will be $\frac{f'''(0)}{3!}$.

$f(x) = e^{2x} \Rightarrow f'(x) = 2e^{2x}$

$\Rightarrow f''(x) = 4e^{2x} \Rightarrow f'''(x) = 8e^{2x}$

Then $f'''(0) = 8$, so the coefficient of x^3 is $\frac{f'''(0)}{3!} = \frac{8}{6} = \frac{4}{3}$.

---

28 E p. 108

Since the inputs are not equally spaced, we must calculate the area of the individual trapezoids.

$\frac{1}{2}(2)[3+k] + \frac{1}{2}(3)[k+9] + \frac{1}{2}(2)[9+11] = 49$

$(3+k) + (\frac{3}{2}k + \frac{27}{2}) + (20) = 49$

$6 + 2k + 3k + 27 + 40 = 98$

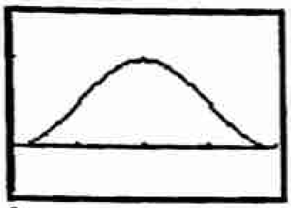
$5k = 25$

$k = 5$

Exam V
Section I
Part B — Calculators Permitted

1. D p. 109

Solution I. Evaluate g(x), sketch its graph, and see on which of the suggested intervals (if any) g displays the desired properties.



$0 \le x \le \pi$
$-.5 \le y \le 1.5$

$$g(x) = \int_0^x \sin(2t)\,dt$$

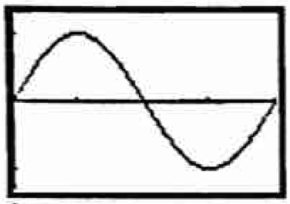
$$= -\frac{1}{2}\cos(2t)\Big]_0^x$$

$$= -\frac{1}{2}(\cos 2x - \cos 0) = \frac{1}{2}(1 - \cos 2x).$$

The curve is decreasing and concave up on the interval $(\frac{3\pi}{4}, \pi)$.

Solution II. Sketch the function $g'(x) = \sin(2x)$.
g is decreasing if $g'(x) \le 0$.
g is concave up if g' is increasing.

In the graph to the right, we see that g' is negative-valued and increasing on the interval $(\frac{3\pi}{4}, \pi)$.



$0 \le x \le \pi$
$-1.3 \le y \le 1.3$

2. B p. 109

Solution I. $x(t) = \sin t \cdot \cos t = \frac{1}{2}\sin(2t)$

Then $v(t) = \cos(2t)$, so $a(t) = -2\sin(2t)$

$$a(t) = 1 \quad \Rightarrow \quad -2\sin(2t) = 1 \quad \Rightarrow \quad \sin(2t) = -\frac{1}{2}$$

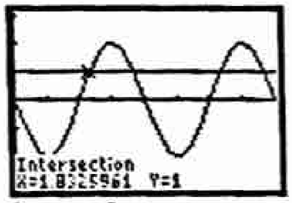
$$\Rightarrow \quad 2t = \frac{7\pi}{6}$$

Then we have $v(t) = \cos(2t) = \cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} \approx -0.866$.

Solution II. The problem can be done graphically.
Define the following functions.

position: Y1 = sin(X)cos(X)
velocity: Y2 = **nDeriv**(Y1,X,X)
acceleration: Y3 = **nDeriv**(Y2,X,X)
Y4 = 1

Graph Y3 and Y4 as shown to the right.
Find their intersection, and then evaluate Y2 at that x-coordinate.
Y2(1.8325961) ≈ −0.866



$0 \le x \le 2\pi$
$-3.1 \le y \le 3.1$

3. B p. 110

Average value $= \dfrac{1}{e-1}\int_1^e x \ln x \, dx$

Solution I. Use a calculator to evaluate this. The result is 1.221 .

Solution II. Determine the exact value of the integral using Integration by Parts.

$$\left\{\begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x & v = \frac{x^2}{2} \end{array}\right\}$$

$$\int x \ln x \, dx = \frac{x^2}{2}\cdot \ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\cdot \ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\cdot \ln x - \frac{x^2}{4} + C$$

Then $\dfrac{1}{e-1}\displaystyle\int_1^e x \ln x \, dx = \dfrac{1}{e-1}\left[\dfrac{x^2}{2}\cdot \ln x - \dfrac{x^2}{4}\right]_1^e$

$$= \frac{1}{e-1}\left[\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right]$$

$$= \frac{e^2+1}{4(e-1)} = 1.221 .$$

---

4. E p. 110

I. The v-shaped graph is an absolute value function. It has a slope of 1 on the right and −1 on the left. Its derivative is the other function. **True**

II. The approximately v-shaped function is decreasing on the left and increasing on the right. The other function is negative-valued on the left and positive-valued on the right. The zero of the second function matches the minimum of the first function. **True**

III. This graph shows both a quartic and a cubic function. The quartic is decreasing to the left of $x = -3.5$, then increasing to $x = 0$, then decreasing to $x = 3.5$, and then increasing again.
The cubic has negative and positive y-values that match those intervals of decreasing and increasing behavior of the quartic. **True**

---

5. B p. 110

I. This says the derivative of f at $x = 2$ is equal to the derivative of f at $x = 5$. Since both derivatives are 0, this is correct. **True**

II. This says the slope of the line joining (2,f(2)) to (5,f(5)) is $\frac{2}{3}$.
The line joining those two points on the curve has a negative slope. **False**

III. $f''(1) < 0$ since the curve is concave down at $x = 1$.
$f''(5) > 0$ since the curve is concave up at $x = 5$. **True**

---

6. B p. 111

$g(x) = f(f(x)) \Rightarrow g'(x) = f'(f(x)) \cdot f'(x)$

Then $g'(1) = f'(f(1)) \cdot f'(1) = f'(2) \cdot f'(1)$

We estimate $f'(2)$ and $f'(1)$ using values in the table.

$$f'(2) \approx \frac{f(2.5) - f(1.5)}{2.5 - 1.5} = \frac{4.4 - 2.4}{1} = 2$$

$$f'(1) \approx \frac{f(1.5) - f(0.5)}{1.5 - 0.5} = \frac{2.4 - 1.8}{1} = 0.6.$$

Then $g'(1) \approx 2 \cdot (0.6) = 1.2$.

7. C p. 111

$x^2 + y^2 = z^2$

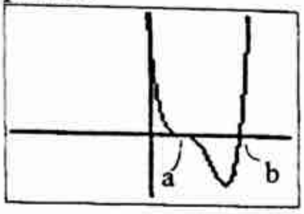
Differentiating implicitly, we have $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

Dividing by 2, and substituting $3\frac{dx}{dt}$ for $\frac{dy}{dt}$ we have $x\frac{dx}{dt} + 3y\frac{dx}{dt} = z\frac{dz}{dt}$.

When x = 3 cm and y = 4 cm, then $z = \sqrt{3^2 + 4^2} = 5$ cm. Also $\frac{dz}{dt} = 2$ cm/sec.

Thus at that moment, we have $3\frac{dx}{dt} + 12\frac{dx}{dt} = 10$, so $\frac{dx}{dt} = \frac{2}{3}$ cm.

8. E p. 111



Since $0 < a < b$, the graph of the position function should look approximately as shown to the left.

I. $\frac{a + b}{2}$ is between a and b. We see from the graph that the particle's position is negative there. **False**

II. The slope of the position function is 0 at time t = a. **True**

III. The position function is increasing at time t = b. **True**

9. A p. 112

$$f(x) = \frac{\ln e^{x+1}}{2x} = \frac{x + 1}{2x}$$

Solution I. $f(x) = 1$ when $x = 1$.

Then $f(1) = 1$, so $g(1) = 1$.

$$f'(x) = \frac{2x \cdot 1 - (x + 1) \cdot 2}{4x^2} = \frac{-2}{4x^2}$$

Then $f'(1) = -\frac{1}{2}$.

Since g is the inverse of f, $g'(1) = \frac{1}{f'(1)} = -2$.

Solution II. Determine the algebraic rule for $g(x) = f^{-1}(x)$.

$$x = \frac{y + 1}{2y} \Rightarrow 2xy - y = 1 \Rightarrow y = \frac{1}{2x - 1}$$

Since $g(x) = \frac{1}{2x - 1}$, then

$$g'(x) = -\frac{2}{(2x - 1)^2}.$$

Hence $g'(1) = -2$.

10. A p. 112

The average rate of change of F over the interval [0,3] is

$$\frac{F(3) - F(0)}{3 - 0} \approx -0.330.$$

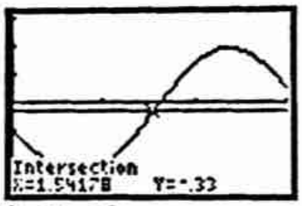
The instantaneous rate of change of F at x is

$F'(x) = -2\sin(2x) - e^{-x}$. We must find where $F'(x) = -0.330$.

We do that graphically by intersecting the graphs of

$Y1 = -2\sin(2x) - e^{-x}$ with $Y2 = -.330$ on the interval [0,3].

The intersection is at $x = 1.542$.



$0 \le x \le 3$

$-3.1 \le y \le 3.1$

---

11. D p. 112

All of the series are continuous, decreasing and positive valued.

(A) and (B) are divergent p-series with $p \le 1$.

Using the integral test on (C), (D), and (E) we obtain:

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \frac{1}{2}\ln(2x+1)\Big|_1^{\infty} = \infty \qquad \text{(C) is divergent}$$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan x\Big|_1^{\infty} = \frac{\pi}{4} \qquad \text{(D) is convergent}$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big|_1^{\infty} = \infty \qquad \text{(E) is divergent}$$

---

12. E p. 113

$$f(x) = f(0) + \int_0^x \frac{1}{\sqrt{1+t^3}}\,dt \quad \text{Since } f(0) = 2, \text{ then } f(4) = 2 + \frac{1}{\sqrt{1+t^3}}\,dt$$

Evaluate the definite integral with a calculator and obtain $f(0) = 2 + 1.8054 = 3.805$

---

13. D p. 113

We need to evaluate $\int_0^5 r(t)\,dt$.

This can be done either with the Fundamental Theorem or with a calculator.

Solution I. Directly with a calculator, we find $\int_0^5 20\,e^{.02t}\,dt \approx 105.$

Solution II.

$$\int_0^5 20\,e^{0.02t}\,dt = 1000\int_0^5 .02\,e^{.02t}\,dt$$

$$= 1000\,e^{.02t}\Big]_0^5 = 1000\,(e^{.1} - 1) \approx 105$$

14. D p. 114

$y(1) = 0$ and $y'(1) = 4 \cdot 1 + 0 = 4$

Then $y(1.5) \approx 0 + 4 \cdot (0.5) = 2.$

$y(1.5) = 2$ and $y'(1.5) = 4 \cdot (1.5) + 2 = 8$

Then $y(2) \approx 2 + 8 \cdot (0.5) = 6$

---

15. A p. 114

The Lagrange error is $|R_n(x)| = \frac{M}{(n+1)!}|x-a|^{n+1}$. Recall that M is the upper bound for all values of the (n+1)st derivative on the given interval , [a, x] in this case.

At x = 2, a = 0, n = 3 and M = 3

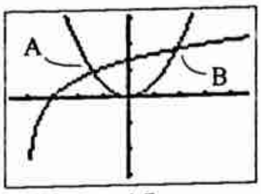
$|R_3(2)| = \frac{3}{4!}|2-0|^4 = \frac{3 \cdot 16}{24!} = 2$

16. B p. 115

First determine the points of intersection of the two curves.
They are at A = –1.3862 and B = 1.8825.
Then evaluate (with a calculator)

$$\int_A^B (\ln(x+4) - .5x^2)\,dx.$$

The value is 3.089.



$-4.7 \le x \le 4.7$
$-3.1 \le y \le 3.1$

---

17. D p. 115

$\int_1^3 f(x)\,dx = \frac{5}{2}$ and $\int_1^5 f(x)\,dx = 10$ $\Rightarrow$ $\int_3^5 f(x)\,dx = 10 - \frac{5}{2} = \frac{15}{2}$

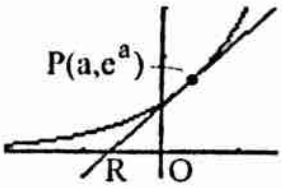
Then $\int_3^5 [2f(x) + 6]\,dx = 2\int_3^5 f(x)\,dx + \int_3^5 6\,dx$

$= 2 \cdot \frac{15}{2} + 6 \cdot 2 = 27$

Exam V
Section II
Part A — Calculators Permitted

1\. p. 117

(a)



$-2 \le x \le 1.5$
$-.5 \le y \le 3$

The slope of the tangent to $y = e^x$ at $x = a$ is $m = e^a$.
Hence the equation of the tangent is

$$y - e^a = e^a(x - a).$$

The x-intercept of the tangent line is found by letting $y = 0$ in the equation above.

$-e^a = e^a(x - a) \quad \Rightarrow \quad -1 = x - a \Rightarrow x = a - 1$

Hence the x-intercept of the tangent is the point $R(a - 1, 0)$.

3: { 1: slope; 1: tangent equation; 1: x - intercept }

(b) The base of the triangle, $\overline{OR}$, has length $|a - 1|$.

The altitude of the triangle is e^a.

Hence the area of ΔPOR is $K = \frac{1}{2}|a - 1|\, e^a$.

2: { 1: base and altitude; 1: triangle area }

(c) With $-1 \le a < 1$, we have $|a - 1| = 1 - a$.

Thus the area, as a function of a, is given by $K(a) = \frac{1}{2}(1 - a)\, e^a$.

$K'(a) = \frac{1}{2}(1 - a)\, e^a + \frac{1}{2} e^a(-1) = \frac{1}{2} e^a(-a)$

The only critical number is $a = 0$.
To verify that there is a maximum value at $a = 0$, we use the Second Derivative Test.

$K''(a) = \frac{1}{2} e^a(-1) + \frac{1}{2}(-a)\, e^a = \frac{1}{2} e^a(-1 - a)$

$K''(0) = \frac{1}{2} \cdot 1 \cdot (-1) < 0$

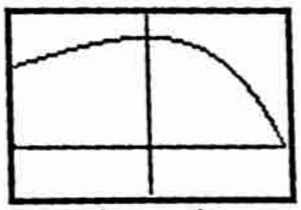
Since $K'(0) = 0$ and $K''(0) < 0$, the function K has a relative maximum at $a = 0$.

4: { 1: derivative; 1: critical number; 1: answer; 1: justication }

The end points are:

$K(-1) = \frac{1}{e} \approx 0.368$ and $K(1) = 0$.

The absolute maximum occurs at $K(0) = \frac{1}{2}$.



$-1 \le x \le 1$
$-.2 \le y \le .6$

2. p. 118

(a) We create a Riemann sum with equal subdivisions.
The thickness of the plate is 1 cm. Each strip has width $\Delta x = 1$ cm.
To calculate the density of each strip, we use the left-hand endpoint as the distance from the y-axis and to compute the height of the strip.

3: { 2: 4 terms; 1: sum }

Strip #	1	2	3	4
Left-hand endpoint	0	1	2	3
Height (cm)	1	1/2	1/3	1/4
Volume (cm^3)	1	1/2	1/3	1/4
Density (g/cm^2)	0	1	4	9
Mass (g)	0	1·(1/2)	4·(1/3)	9·(1/4)

A Riemann sum for the mass of the plate is the sum of the masses of the four strips:

$$R_4 = 0 + 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} + 9 \cdot \frac{1}{4} = \frac{49}{12}.$$

(b) If we have a strip of width Δx occurring at coordinate c_k, then

4: { 1: limits; 3: integrand }

the height of the strip will be $\frac{1}{1 + c_k}$;

the volume of the strip will be $\frac{1}{1 + c_k} \cdot 1 \cdot \Delta x$;

the density of the strip will be c_k^2.

Then the mass of that strip will be $c_k^2 \cdot \frac{1}{1 + c_k} \cdot 1 \cdot \Delta x$.

Adding the masses of all the strips, we obtain a Riemann sum for the total mass of the plate:

$M = \sum_{k=1}^{n} c_k^2 \cdot \frac{1}{1 + c_k} \Delta x$. As $n \to \infty$ and the norm of the partition approaches 0, the limit of the Riemann sum, is $\int_0^4 \frac{x^2}{1 + x}\, dx$.

(c)

2: { 1: integral; 1: answer }

Using a calculator approximation, $\int_0^4 \frac{x^2}{1 + x}\, dx = 5.609$.

Hence the mass of the metal plate is 5.609 grams.

Exam V
Section II
Part B — No Calculators

3. p. 119

(a)

Since the n^{th} derivative of f at $x = 2$ is given by $f^{n}(2) = \frac{n!}{n \cdot 3^n}$, and $f(2) = 0$, we have:

$f'(2) = \frac{1}{3}$, $f''(2) = \frac{1}{9}$, $f'''(2) = \frac{2}{27}$; $a_0 = 0$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{18}$, $a_3 = \frac{1}{81}$;

$T_3(x) = \frac{1}{3}(x-2) + \frac{1}{18}(x-2)^2 + \frac{1}{81}(x-1)^3$

4: { 1: polynomial about x = 2; 3: $P_3(x)$ − 1 for each incorrect term }

(b) We use the Ratio Test.

$a_n = \frac{(x-2)^n}{n \cdot 3^n}$, so $a_{n+1} = \frac{(x-2)^{n+1}}{(n+1) \cdot 3^{n+1}}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}}}{\frac{(x-2)^n}{n \cdot 3^n}}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n}\right| =$$

$$\left|\frac{x-2}{3}\right| \cdot \frac{n}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-2}{3}\right| \cdot \lim_{n\to\infty}\frac{n}{n+1} = \left|\frac{x-2}{3}\right| \cdot 1 = \left|\frac{x-2}{3}\right|$$

We have convergence when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

Therefore we need $\left|\frac{x-2}{3}\right| < 1$; that is equivalent to $|x-2| < 3$, so we have our initial interval of convergence as the open interval $(-1,5)$.

Now test the endpoints of the interval.

When $x = 5$, the series is $\sum_{n=1}^{\infty}\frac{(5-2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$. This is the harmonic series, which diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty}\frac{(-1-2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$. This is the opposite of the alternating harmonic series, which converges by the Alternating Series Test.

The complete interval of convergence is all x for which $-1 \le x < 5$.

5: { 1: set up ratio; 1: computes limit of ratio; 1: identifies interval of convergence; 1: considers both endpts; 1: analysis / conclusion for both endpoints }

4. p. 120

(a) $y^3 + y^2 - 5y - x^2 = -4$

Differentiate implicitly.

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

2: { 1: implicit diff; 1: solve for $\frac{dy}{dx}$ }

(b) At the point (2,0), $\frac{dy}{dx} = \frac{4}{-5}$.

Then the equation of the tangent line there is $y - 0 = -\frac{4}{5}(x - 2)$.

2: { 1: slope; 1: tangent equation }

(c) There is a horizontal tangent line if $\frac{dy}{dx} = 0$. This occurs if $x = 0$.

Vertical tangents occur at points that make the denominator of the expression for $\frac{dy}{dx}$ equal to 0.

That is, $3y^2 + 2y - 5 = 0$

$(3y + 5)(y - 1) = 0$

$y = -\frac{5}{3}$ or $y = 1$

Start with $y = -\frac{5}{3}$. Returning to the original equation, we have

$$-\frac{125}{27} + \frac{25}{9} + \frac{25}{3} - x^2 = -4.$$

Then $x^2 = \frac{283}{27}$, so $x = \pm\sqrt{\frac{283}{27}}$.

Using $y = 1$, we have $1 + 1 - 5 - x^2 = -4$.

Then $x^2 = 1$, so $x = \pm 1$.

Hence there are four points where vertical tangents occur:

$\left(\sqrt{\frac{283}{27}}, -\frac{5}{3}\right), \left(-\sqrt{\frac{283}{27}} -\frac{5}{3}\right)$, (1,1) and (−1,1).

5: {
horizontal tangent
1: sets numerator of $\frac{dy}{dx}$ equal to 0
1: point
vertical tangent
1: sets denominator c $\frac{dy}{dx}$ equal to 0
2: points
}

5. p. 121

(a) At $x = 10$, f has a relative maximum.
f is increasing for $0 \le x \le 10$, since $f'(x) \ge 0$ on that interval.
f is decreasing for $10 \le x \le 15$, since $f'(x) \le 0$ on that interval.
Thus f has a relative maximum at $x = 10$.
There is no other relative maximum in the open interval $(0,20)$ since f is increasing on the interval $[15,20]$.

2: { 1: answer; 1: justification }

(b) The graph of f is concave down if f' is decreasing.
This happens on the interval $(5,12.5)$.

2: { 1: interval; 1: justification }

(c) $f'(x)$ exists for all x in the interval $[0,20]$, so f is continuous on that interval.

1. On the interval $[0,5]$, f' is linear and positive. Hence f is quadratic and increasing on that interval.
   The area under f' is 25 on $[0,5]$. Since $f(0) = 10$, then $f(5) = 35$.
   $f'(0) = 0$, so this first quadratic piece of f has a horizontal tangent at $x = 0$.

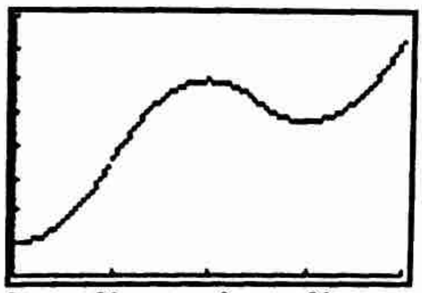
2. On the interval $[5,12.5]$, $f'(x)$ is linear, changing from positive- to negative-valued at $x = 10$. Hence f is quadratic, increasing on $[5,10]$ and decreasing on $[10,12.5]$.
   The area under f' from $x = 5$ to $x = 10$ is 25. Since $f(5) = 35$, then $f(10) = 60$.
   The area between the x-axis and the graph of f' on the interval $[10,12.5]$ is 6.25. Since $f'(x)$ is negative on that interval and $f(10) = 60$, then $f(12.5) = 53.75$.

3. On the interval $[12.5,20]$, $f'(x)$ is linear, changing from negative- to positive-valued at $x = 15$. Hence f is quadratic, decreasing on $[12.5,15]$ and increasing on $[15,20]$.
   The area between the x-axis and the graph of f' on the interval $[12.5,15]$ is 6.25. Since $f'(x)$ is negative on that interval and $f(12.5) = 53.75$, then $f(15) = 47.5$.
   The area under f' from $x = 15$ to $x = 20$ is 25. Since $f(15) = 47.5$, then $f(20) = 72.5$.

5: { 1: critical points; 1: inflection points; 3: graph }

The graph of f on the interval $[0,20]$ is shown to the right.
The critical points occur at $(10,60)$ and at $(15,47.5)$. Those correspond to points where $f'(x)$ changes sign.

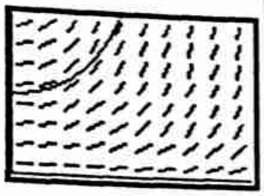
The inflection points are at $(5,35)$ and $(12.5,53.75)$. Those correspond to points where the increasing/decreasing nature of f' changes.



$0 \le x \le 20$ $\quad$ $0 \le y \le 80$

6. p. 122

(a) Starting at the point (0,2), we follow the curve with slopes indicated in the direction field.



1: solution curve

(b) Start with the given differential equation: $\frac{dy}{dx} = .5\,xy$

Separate variables: $\frac{dy}{y} = \frac{1}{2}x\,dx$

Integrate: $\ln|y| = \frac{1}{4}x^2 + C$

Use the point (0,2): $\ln 2 = C$

Rewrite the solution: $\ln|y| = \frac{1}{4}x^2 + \ln 2$

$$y = e^{x^2/4 + \ln 2} = e^{\ln 2} \cdot e^{x^2/4}$$

$$y = 2e^{x^2/4}$$

4: { 1: separate variables; 1: antidifferentiate; 1: use initial condition; 1: solve for y }

(c) Using $x = 2$ in the solution obtained in part (b), we have $y(2) = 2e$.

1: $y(2)$

(d) $y(0) = 2$ and $y'(0) = .5(0)(2) = 0$

Then $y(1) \approx 2 + 0 \cdot 1 = 2$.

$y(1) = 2$ and $y'(1) = .5(1)(2) = 1$

Then $y(2) \approx 2 + 1 \cdot 1 = 3$.

The true value obtained in part (c) is $y(2) = 2e$.

Since $e \approx 2.7$ and $2e \approx 5.4$, the estimate obtained with Euler's Method is low by 2.4, a large error of due to the large step size.

3: { 1: Euler's method; 1: approximation; 1: comparsion }

Exam VI
Section I
Part A — No Calculators

1. A p. 123

$$\int\sqrt{4-2x}\,dx = \int(4-x)^{1/2} = -\tfrac{1}{2}\int(-2)(4-x)^{1/2}\,dx$$

$$= -\frac{1}{2}\cdot\frac{2}{3}(4-2x)^{3/2} + C = -\frac{1}{3}(4-2x)^{3/2} + C$$

2. E p. 123

$$s(t) = \int_1^t (\sqrt{x} - x + 1)\,dx$$

$$v(t) = s'(t) = \sqrt{t} - t + 1$$

$$a(t) = v'(t) = \frac{1}{2\sqrt{t}} - 1$$

Then $a(t) = 0$ when $\frac{1}{2\sqrt{t}} - 1 = 0$. This implies that $1 = 2\sqrt{t}$, so $\sqrt{t} = \frac{1}{2}$, and $t = \frac{1}{4}$.

$v(\frac{1}{4}) = \frac{1}{2} - \frac{1}{4} + 1 = \frac{5}{4}$ m/s.

3. B p. 124

$$\frac{dy}{dx} = \frac{\cos x}{e^y} \Rightarrow e^y\,dy = \cos x\,dx$$

$$\Rightarrow \int e^y\,dy = \int \cos x\,dx$$

$$\Rightarrow e^y = \sin x + C$$

$$\Rightarrow y = \ln(\sin x + C)$$

Then $y(0) = \ln(\sin 0 + C) = \ln C$.

We are given the initial condition that $y(0) = 0$, so $C = 1$. Hence $y = \ln(\sin x + 1)$.

Then $y(\frac{\pi}{2}) = \ln(\sin\frac{\pi}{2} + 1) = \ln 2$.

4. E p. 124

I. Since $h'(x) > 0$ on the interval $(1,2)$, then h is increasing on that interval. **False**

II. The derivative of h is positive-valued to the left of $x = 2$ and negative-value to the right of $x = 2$. Hence h is increasing to the left of $x = 2$ and decreasing to the right of $x = 2$. The function h has a local maximum at $x = 2$. **True**

III. From the graph, we read that $h'(1) = 2$. Then an equation of the tangent line at $(1,-1)$ is $y + 1 = 2(x - 1)$. This simplifies to $y = 2x - 3$. **True**

5. E p. 124

This is an improper integral.

$$\int_1^{\infty} \frac{x}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{x}{1+x^2}\,dx = \lim_{b\to\infty} \frac{1}{2}\ln(1+x^2)\Big]_1^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln b - \frac{1}{2}\ln 1\right).$$

This limit does not exist.

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6. D p. 125

Solution I. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -2\cos t \sin t$. Hence $\frac{dy}{dx} = \frac{-2\cos t \sin t}{\cos t} = -2\sin t$.

To compute $\frac{d^2y}{dx^2}$, we need $\frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{-2\cos t}{\cos t} = -2.$

The second derivative is constant, not depending $t = \frac{\pi}{2}$.

Solution II. We eliminate the parameter. $x^2 = \sin^2 t$ and $y = \cos^2 t$,
so $x^2 + y = 1$. Hence $y = 1 - x^2$. Then $\frac{dy}{dx} = -2x$ and $\frac{d^2y}{dx^2} = -2$.

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7. D p. 125

I. When $P = 600$, $\frac{dP}{dt} = 0$. There is no growth in the fish population. False

II. If $P > 600$, then the factor $(1 - \frac{P}{600})$ is negative, so $\frac{dP}{dt} < 0$. True

III. The carrying capacity in this model is $P = 600$. True

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8. C p. 126

The derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$.

Thus $\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$.

Hence $\int_3^5 f'(g(x)) \cdot g'(x)\,dx = f(g(5)) - f(g(3))$.

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9. D p. 126

Divide the numerator and denominator of the complex fraction by x.

$$\lim_{x\to\infty} \frac{x - \frac{1}{2x}}{2x - \frac{1}{6x}} = \lim_{x\to\infty} \frac{1 - \frac{1}{2x^2}}{2 - \frac{1}{6x^2}}$$

The fractions in the numerator and denominator have limit 0 as $x \to \infty$.

Hence the limit of the original complex fraction is $\frac{1}{2}$.

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10. A p. 126

$\frac{dy}{dx} = \frac{(4x-3)\cdot 3 - (3x+4)\cdot 4}{(4x-3)^2}$. When $x = 1$, $\frac{dy}{dx} = \frac{1\cdot 3 - 7\cdot 4}{1} = -25$.

The only proposed answer with a slope of -25 is the first one.

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11. C p. 127

$\frac{dy}{dx} = \sqrt{x} \quad \Rightarrow \quad y = \frac{2}{3}x^{3/2} + C$

We need to compute $\frac{y(4) - y(0)}{4 - 0}$. $\frac{y(4) - y(0)}{4 - 0} = \frac{\frac{2}{3} \cdot 8}{4} = \frac{4}{3}$

---

12. C p. 127

$f(x) > 0$ for $-1 < x < 1$ and $f(x) < 0$ for $1 < x < 2$.
Since $f(x) = g'(x)$, this means that g is increasing on $(-1,1)$ and decreasing on $(1,2)$. Thus g has its maximum value at $x = 1$. $g(1) = \int_1^1 f(t)\,dt = 0$.

---

13. B p. 127

The polar curve $r = 1 + \cos\theta$ is a cardioid. The part produced as θ goes from 0 to π encloses the same area as that generated as θ goes from π to 2π.

Although we would usually compute the area as $A = \frac{1}{2}\int_0^{2\pi}(f(\theta))^2\,d\theta$, in this case we can use the interval $[0,\pi]$ instead of $[0,2\pi]$ and eliminate the factor $\frac{1}{2}$.

Thus $A = \int_0^{\pi}(1 + \cos\theta)^2\,d\theta$.

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14. D p. 128

$a(t) = \cos t \quad \Rightarrow \quad v(t) = \sin t + C$
$v(0) = 2 \quad \Rightarrow \quad v(t) = \sin t + 2$

The velocity function is never negative. The particle always moves to the right.

Hence the total distance traveled from $t = 0$ to $t = \frac{\pi}{2}$ is

$$D = \int_0^{\pi/2}(\sin t + 2)\,dt = [-\cos t + 2t]_0^{\pi/2} = \pi + 1$$

---

15. B p. 128

$\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2\cos t$. Then $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 4\sin^2 t + 4\cos^2 t = 4$.

The arc length, as t varies from $t = 0$ to $t = 2$, is $L = \int_0^2 \sqrt{4}\,dt = \int_0^2 2\,dt = 4$.

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16. A p. 128

$\sum_{k=1}^{\infty}(\frac{1}{3})^{2k} = \sum_{k=1}^{\infty}(\frac{1}{9})^k$. This is a geometric series with first term $a = \frac{1}{9}$ and common ratio $r = \frac{1}{9}$. It converges to $S = \frac{1/9}{1 - \frac{1}{9}} = \frac{1}{9 - 1} = \frac{1}{8}$.

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17. D p. 129

The curve is a version of the reciprocal function and has symmetry around the line $y = x$. The points on the curve closest to the origin are the two points (2,2) and (–2,–2). Each of these is $2\sqrt{2}$ units from the origin.

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18. C p. 129

The given limit is one of the definitions of $f'(a)$. Thus we know that $f'(a) = 0$.

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19. B p. 129

$v(t) = (2t, \sin(t)) \quad \Rightarrow \quad s(t) = (t^2 + C, -\cos(t) + D)$.
$s(0) = (C, -1 + D)$. Since we are given that $s(0) = (0, 1)$,
we determine that $C = 0$ and $D = 2$.
Then $s(t) = (t^2, 2 - \cos(t))$. We then compute $s(\pi) = (\pi^2, 2 - \cos(\pi)) = (\pi^2, 3)$

---

20. D p. 130

The total number of cars in the twenty-mile stretch is given by

$$\int_0^{20} (500 + 100\sin(\pi x))\,dx = \left[500x - \frac{100}{\pi}\cos(\pi x)\right]_0^{20}$$

$$= \left(10000 - \frac{100}{\pi}\right) - \left(0 - \frac{100}{\pi}\right) = 10{,}000$$

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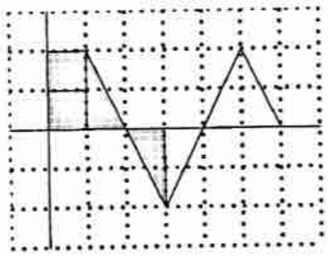
21. D p. 130

$G'(x) = f(x) \quad \Rightarrow \quad G'(3) = -2.$
$G(3)$ is evaluated by counting up areas. The negative-valued region on [2,3] cancels out the positive-valued region on [1,2].
Hence $G(3) = 2$.
Then a linear approximation for G near $x = 2$ is given by
$y - 2 = -2(x - 3)$. Hence $y = -2x + 8$.



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22. C p. 130

To determine the radius of convergence, we evaluate $\lim_{k \to 0} \left| \frac{a_{k+1}}{a_k} \right|$.

$$\lim_{k \to 0} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to 0} \left| \frac{3^{k+1}(x+2)^{k+1}}{k+2} \cdot \frac{k+1}{3^k (x+2)^k} \right|$$

$$= 3 \cdot |x + 2| \cdot \lim_{k \to 0} \frac{k+1}{k+2} = 3 \cdot |x + 2|.$$

The series converges if $\lim_{k \to 0} \left| \frac{a_{k+1}}{a_k} \right| < 1$.

Hence we must have $3 \cdot |x + 2| < 1$, or $|x + 2| < \frac{1}{3}$.

This gives an interval centered at $x = -2$ with a radius $r = \frac{1}{3}$.

The radius of convergence is $r = \frac{1}{3}$.

---

23. B p. 131

$V = \int_0^\infty \pi (e^{-2x})^2 dx = \pi \int_0^\infty e^{-4x} dx$. This is an improper integral.

$$\pi \int_0^\infty e^{-4x} dx = \lim_{b \to \infty} \pi \int_0^b e^{-4x} dx = \lim_{b\to\infty} \left[-\frac{\pi}{4} e^{-4x} \right]_0^b = \lim_{b\to\infty} \left(-\frac{\pi}{4} e^{-4b} \right) + \frac{\pi}{4} = \frac{\pi}{4} \text{ units}^3$$

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24. D p. 131

I. The slopes level out at $y = 1$. Hence there is no way to get from $(-2,0)$ to $(0,2)$ on the same solution curve. False

II. The slopes level out at $y = 1$. True

III. The slopes are constant for a fixed value of y. True

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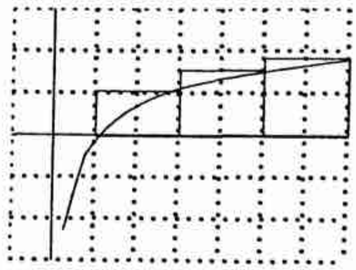
25. C p. 131

The width of each rectangle is 2.
The heights of the rectangles are, respectively, ln 3, ln 5, and ln 7.

The approximation is $2(\ln 3 + \ln 5 + \ln 7)$.



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26. C p. 132

I. $f(2) \approx 2.8$. $f'(2) = 0$ True

II. $\int_0^1 f(x)\,dx > 0$. Since the graph of f is concave down at $x = 2$, $f''(2) < 0$. True

III. Letting $g(x) = 1 - x + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{15} - \ldots$ be the given Maclaurin series, we find $g(0) = 1 = f(0)$.
However, $g'(0) = -1 \neq f'(0)$, since f is increasing at $x = 0$. False

---

27. C p. 132

At $(0,2)$, $\frac{dy}{dx} = 0$. Then the first step of Euler's method takes us to $(0.5,2)$.
At $(0.5,2)$, $\frac{dy}{dx} = \frac{10 \cdot 0.5}{0.5 + 2} = 2$. In the second step of Euler's method, y changes by $0.5 \cdot 2 = 1$ (to 3).

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28. A p. 132

$y = x \ln x \Rightarrow y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1 \Rightarrow y'' = \frac{1}{x} \Rightarrow y''' = -\frac{1}{x^2}$

Continuing, we have $y^{(4)} = \frac{2}{x^3} \Rightarrow y^{(5)} = -\frac{6}{x^4} \Rightarrow y^{(5)}(1) = -6$

The desired coefficient is $\frac{-6}{5!} = \frac{-6}{120} = -\frac{1}{20}$.

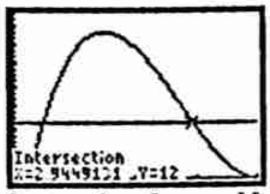
Exam VI
Section I
Part B — Calculators Permitted

1. E p. 133

Ave. vel. = $\frac{x(4) - x(2)}{4 - 2} = \frac{0 - (3 \cdot (-8))}{2} = 12$

Inst. vel. = $(t-4)^3 + 3(t-4)^2(t+1)$

Use a calculator to find the intersection in the interval [2,4] of the functions $Y_1 = (x-4)^3 + 3(x-4)^2(x+1)$ and $Y_2 = 12$. The intersection is at $x = 2.94$.



$0 \le x \le 4 \quad 0 \le y \le 35$

2. B p. 133

$f'(x) = 4x^3 + 2ax + 8 \qquad f''(x) = 12x^2 + 2a$

At the point of inflection, $f''(x) = 12x^2 + 2a = 0$, so $a = -6x^2$.

Substitute this in the equation for $f'(x)$ to obtain $f'(x) = 4x^3 - 12x^3 + 8 = 0$.

Hence $8x^3 = 8$, so $x = 1$. Then $a = -6$.

3. C p. 134

I. $f''(x) > 0$ on the interval (1,3), so f is concave up there. — False

II. $f''(x) > 0$ on both the left and the right of $x = 1$. The concavity of the graph of f doesn't change there. — False

III. The derivative of f' is positive-valued on (2,3), so f' is increasing on that interval. If $f'(2) = 0$, then $f'(3) > 0$. That means that the function f is increasing at $x = 3$. — True

4. B p. 134

Using the graphing calculator, we graph $y = 2\sin x$ and $y = e^{x/2}$. Next determine the intersection points $A = 0$ and $B = 2.04464$

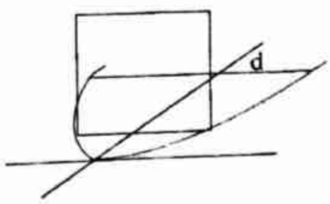
The area under $y = 2\sin x$ and above $y = e^{x/2}$ is given by

$$\int_A^B \left(2\sin x - e^{x/2}\right)dx$$

The value is 1.398.

5. B p. 134

The volume of the solid is $\int_0^d (2x)^2 dy = \int_0^d 4x^2\,dy$.



Since $y = x^2$, this is $V = \int_0^d 4y\,dy = 2d^2$.

If the volume is 72, then $2d^2 = 72$, so $d = 6$.

6. C p. 135

$g(x) = \int_{\pi/2}^{x} \cos t\,dt = \sin x - \sin\frac{\pi}{2} = \sin x - 1.$

Since the interval $[-\pi, \pi]$ includes one complete period, we need the maximum value of the function $g(x) = \sin x - 1$. That maximum value is 0.

7. B p. 135

If $r = \sin\theta$, we multiply both sides by r to obtain $r^2 = r\sin\theta$.
To convert to Cartesian coordinates, we replace r^2 with $x^2 + y^2$ and $r\sin\theta$ with y. This gives $x^2 + y^2 = y$.

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8. D p. 135

The following two limits must exist: $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} f'(x)$.
Set left-hand and right-hand limits for each of these equal.
For $f(x)$: $e^0 + 3 = 0a + b$
For $f'(x)$: $-e^0 = a$
Thus $a = -1$ and $b = 4$. Adding, we have $a + b = 3$.

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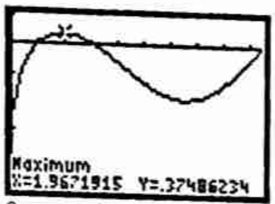
9. B p. 136

For the function f to be increasing and concave down, its derivative f' must be positive-valued and decreasing.
Graph $f'(x)$ and note that the function has the two necessary properties on an interval starting at $x = 1.97$. The correct answer is $(1.97, 3.14)$.



$0 \le x \le 3\pi \quad -5 \le y \le 1$

---

10. D p. 136

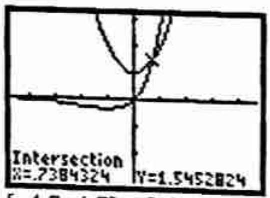
The series $\sum_{n=1}^{\infty} \frac{1}{n^{a-27}}$ is a p-series with $p = a - 27$.

Such a series converges only if $p > 1$. Hence we must have $a - 27 > 1$.
Thus $a > 28$; the least integer value that works is $a = 29$.

---

11. B p. 136

The given Maclaurin series is that for the function $f(x) = x\,e^x$.
We use a calculator to find the intersection of the curves $Y_1 = x\,e^x$ and $Y_2 = 1 + x^2$.
The intersection is at $x = 0.738$.



$[-4.7, 4.7] \times [-3.1, 3.1]$

---

12. A p. 137

We differentiate $x^2 + y^2 = 4$ implicitly: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.

Substituting the given values for x, y, and $\frac{dx}{dt}$ gives $2 \cdot 1 \cdot 2 + 2 \cdot \sqrt{3} \cdot \frac{dy}{dt} = 0$.

Then $\frac{dy}{dt} = -\frac{2}{\sqrt{3}}$.

---

13. B p. 137

Continuous compounding at 4% annual interest is modeled by the function $B(t) = B_0 e^{0.04t}$. (This is the function that satisfies the given differential equation.)
If when $t = 5$, the balance B is to be 4000, we have the equation $4000 = B_0 e^{0.2}$.
Hence $B_0 = \frac{4000}{e^{0.2}} \approx 3274.92$. The nearest answer choice is \$3300.

---

14. B p. 137

At $x = 1$, $f(x) = 0$ (The curve is cutting the x-axis.)
$f'(x) < 0$ (The function f is decreasing.)
$f''(x) > 0$ (The graph of f is concave up.)

Thus $f'(1) < f(1) < f''(1)$.

---

15. C p. 138

This looks like Integration by Parts. Let $u = f(x)$ and $v' = \sec^2 x$.
Then $u' = f'(x)$ and $v = \tan x$.

That gives $\int f(x)\sec^2 x\,dx = f(x)\tan x - \int f'(x)\tan x\,dx$.

Comparing this with the given relationship, we find that $f'(x) = 9x^2$.

Then $f(x)$ could be $3x^3$.

---

16. D p. 138

We substitute for $\sqrt{x}$ and for dx, and we change the limits of integration.

$\sqrt{x} = u - 1$

$u = \sqrt{x} + 1 \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow \quad \frac{1}{\sqrt{x}}dx = 2\,du$

$x = 1$ fi $u = 2$
$x = 4$ fi $u = 3$

The integrand $\frac{1}{\sqrt{x}\,(\sqrt{x} + 1)}dx$ becomes $\frac{1}{u}2\,du$.

The transformed integral is $\int_2^3 \frac{2}{u}\,du$.

---

17. D p. 138

I. Use l'Hôpital's Rule.

$$\lim_{x\to 2}\frac{f(x)}{\sin(x-2)} = \lim_{x\to 2}\frac{f'(x)}{\cos(x-2)} = \frac{1}{1} = 1$$ **True**

II. Use l'Hôpital's Rule.

$$\lim_{x\to 1}\frac{f(x-1)}{f(x+1)} = \lim_{x\to 1}\frac{f'(x-1)}{f'(x+1)} = \frac{0}{1} = 0$$ **False**

III. Use l'Hôpital's Rule.

$$\lim_{x\to 2}\frac{[f(x)]^2}{(x-2)^2} = \lim_{x\to 2}\frac{2f(x)\cdot f'(x)}{2(x-2)} = \lim_{x\to 2}\frac{f(x)\cdot f'(x)}{(x-2)}$$

$$= \lim_{x\to 2}\frac{f'(x)f'(x) + f(x)f''(x)}{1} = \frac{1\cdot 1 + 0\cdot 0}{1} = 1$$ **True**

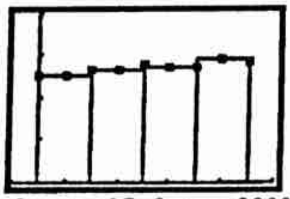
Exam VI
Section II
Part A — Calculators Permitted

1. p. 140

(a) Shown to the right are the four rectangles for a midpoint approximation with four subdivisions.

$M_4 = 6[1268 + 1316 + 1369 + 1451]$
$= 32{,}424$ kilowatts.

The kilowatt usage of power recorded by this substation for those 24 hours was approximately 32,424 kilowatts.



$-3 \le x \le 27 \quad 0 \le y \le 2000$

$3: \begin{cases} 1: \text{midpoint method} \\ 1: \text{answer} \\ 1: \text{explanation} \end{cases}$

(b) Rate of change $\approx \dfrac{P(15) - P(9)}{15 - 9} = \dfrac{1369 - 1316}{6} = \dfrac{53}{6} \approx 8.833$ K/hr^2.

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer (with units)} \end{cases}$

(c) Ave. rate of power usage $= \dfrac{1}{24}\displaystyle\int_0^{24} f(t)\,dt = \dfrac{1}{24}\int_0^{24}\left(1245 + 10t\,e^{0.25\cos t}\right) dt$

We use a calculator to evaluate this integral. Ave. rate of power usage = 1364.478 K/hr.

$4: \begin{cases} 1: \text{limits} \\ 2: \text{integrand + constant} \\ 1: \text{answer (with units)} \end{cases}$

---

2. p. 141

(a) $g\left(\frac{1}{2}\right) \approx 0.210$

1: answer

(b) $g'(x) = \cos\left(e^{x/2}\right);\ g'(0) = \cos 1 \approx 0.540$

1: answer

(c) $x \approx 0.903$

$g'(x)$ changes from positive to negative at x = 0.903

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) $x = 2.289$ and $x = 3.676$

on $(-1, 2.289), g''(x) < 0$

on $(2.289, 3.676), g''(x) > 0$

on $(3.767, 4), g''(x) < 0$

Therefore, at $x = 2.289$ and $x = 3.676$, there are inflection points.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(e) $g(0.903) \approx 0.269$ is the maximum of g on the interval [0, 4].

The absolute maximum must occur at $x = 0.903$ or at an endpoint.

$g(0) \approx 0$

$g(0.903) \approx 0.269$

$g(4) \approx -0.455$

So the absolute maximum value of g on the closed interval [0, 4] is 0.269.

$3: \begin{cases} 1: \text{interior analysis} \\ 1: \text{endpoint analysis} \\ 1: \text{answer} \end{cases}$

Exam VI
Section II
Part B — No Calculators

3. p. 142

(a) We calculate slopes at each of the twelve points.

At (0,1), m = –1	At (0,–1), m = 1.
At (0,2), m = –1/2.	At (0,–2), m = 1/2.
At (1,1), m = 0.	At (1,–1), m = 0.
At (1,2), m = 0.	At (1,–2), m = 0.
At (2,1), m = 1.	At (2,–1), m = –1.
At (2,2), m = 1/2.	At (2,–1), m = –1/2.

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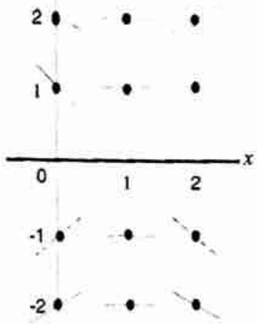
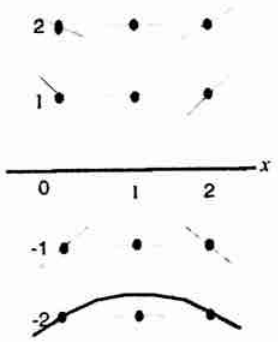


Then draw short line segments through each of the points with the appropriate slope.

2 : { 1: zero slopes; 1: nonzero slopes }

(b)



1: curve through (0,–2)

(c) $y''(x) = \dfrac{y\cdot 1-(x-1)\cdot y'}{y^2}$

$= \dfrac{y\cdot 1-(x-1)\cdot\left(\dfrac{x-1}{y}\right)}{y^2}$

$= \dfrac{y^2-(x-1)^2}{y^3}.$

At $(0,-2)$, $y''(0) = \dfrac{4-(-1)^2}{(-2)^3} = -\dfrac{3}{8}$

2: { 1: $y''(x)$; 1: answer }

(d) $\int y\dfrac{dy}{dx} = \int (x-1)\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} - x + C$

$y(0) = -2 \Rightarrow C = 2$

$y^2 = x^2 - 2x + 4$

$|y| = \sqrt{x^2-2x+4}$. Since the initial condition has $y < 0$,

choose $y = -\sqrt{x^2-2x+4}$.

4: { 1: separates variables; 1: antiderivatives; 1: uses initial conditions; 1: solves for y }

4. p. 143

(a) We integrate in order to find x and y as functions of t.

$x = \int \frac{1}{2\sqrt{t+1}}\,dt = \sqrt{t+1} + C$. Since $x(0) = 1$, we determine that $C = 0$.

$y = \int \frac{1}{(t+1)^2}\,dt = -\frac{1}{t+1} + D$. Since $y(0) = 0$, we determine that $D = 1$.

Hence, we have the parametric equations $\begin{cases} x(t) = \sqrt{t+1}\ ; \\ y(t) = -\frac{1}{t+1} + 1\ . \end{cases}$

4: $\begin{cases} 2\text{: antiderivatives} \\ 1\text{: use initial conditions} \\ 1\text{: answer} \end{cases}$

(b) $x^2 = t + 1$, so $y = -\frac{1}{x^2} + 1$, for $x \geq 1$ because $t \geq 0$.

1: $y = f(x)$

(c) When t varies from 3 to 15, x varies from 2 to 4 while y varies from $\frac{3}{4}$ to $\frac{15}{16}$.

The average rate of change of y with respect to x is $\frac{\frac{15}{16} - \frac{3}{4}}{4 - 2} = \frac{3}{32}$.

2: $\begin{cases} 1\text{: difference quotient} \\ 1\text{: answer} \end{cases}$

(d) Since in (b) we found $y = -\frac{1}{x^2} + 1$, we have $\frac{dy}{dx} = \frac{2}{x^{-3}}$.

When $t = 8$, then $x = 3$. Hence $\frac{dy}{dx} = \frac{2}{27}$.

2: $\begin{cases} 1\text{: } \frac{dy}{dx} \\ 1\text{: answer} \end{cases}$

---

5. p. 144

(a) $y^2 - y + e^x = \cos x \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} + e^x = -\sin x$

$\Rightarrow \frac{dy}{dx}(2y - 1) = -e^x - \sin x \Rightarrow \frac{dy}{dx} = \frac{e^x + \sin x}{1 - 2y}$

2: $\begin{cases} 1\text{: implicit diff} \\ 1\text{: solves for } \frac{dy}{dx} \end{cases}$

(b) At the point (0, 1), $\frac{dy}{dx} = \frac{1+0}{1-2} = -1$.

Hence $y - 1 = -1(x - 0)$, or $y = 1 - x$.

2: $\begin{cases} 1\text{: slope} \\ 1\text{: tangent equation} \end{cases}$

(c) Start with the expression $\frac{dy}{dx}(2y - 1) = -e^x - \sin x$.

Differentiating implicitly with respect to x, we have

$\frac{d^2y}{dx^2}\cdot(2y - 1) + \frac{dy}{dx}\cdot 2\frac{dy}{dx} = -e^x - \cos x$.

At the point (0,1), $\frac{dy}{dx} = -1$, so we have $\frac{d^2y}{dx^2}\cdot(2 - 1) + 2(-1)^2 = -1 - 1$.

Therefore, $\frac{d^2y}{dx^2} = -4$.

2: $\begin{cases} 1\text{: } \frac{d^2y}{dx^2} \\ 1\text{: answer} \end{cases}$

(d) Since the tangent line is $y = 1 - x$, when $x = 0.5$, then $y = 0.5$ as well.

1: estimation

(e) The value from the tangent line approximation is larger than the actual function value. In (c), we find that the second derivative is negative-valued at (0,1), so the curve is concave down in the vicinity of that point. That means that the tangent line is above the curve.

2: $\begin{cases} 1\text{: answer} \\ 1\text{: explanation} \end{cases}$

6. p. 145

(a) Rewrite the function as $f(x) = \dfrac{1/4}{1 + (x/2)^2}$.

This is a geometric series with first term $a = \frac{1}{4}$ and common ratio $R = -\left(\frac{x}{2}\right)^2$.

Hence $f(x) = \frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64} - \frac{x^6}{256} + \ldots + (-1)^n \frac{x^{2n}}{2^{2n+2}} + \ldots$.

2: { 1: first four terms; 1: general term }

(b) A geometric series converges if and only if the common ratio satisfies $|R| < 1$. Hence we must have $\left|\frac{x^2}{4}\right| < 1$. Then $|x| < 2$. The radius of convergence is $R = 2$.

3: { 1: sets up ratio; 1: limit; 1: applies ratio test to conclude radius of convergence is 2 }

(c)

$$g(x) = \int_0^x f(t)\,dt = \int_0^x \left[\frac{1}{4} - \frac{t^2}{16} + \frac{t^4}{64} - \frac{t^6}{256} + \cdots (-1)^n \frac{t^{2n}}{2^{2n+2}} + \cdots\right] dt$$

$$= \frac{x}{4} - \frac{x^3}{3 \cdot 16} + \frac{x^5}{5 \cdot 64} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)\cdot 2^{2n+2}} + \ldots$$

2: { 1: antiderivative; 1: general term }

(d) At $x = 2$, $g(2) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)2^{n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)}$

2: { 1: antiderivative; 1: conclusion }

which converges by the Alternating Series Test.

$$g(x) = \int_0^x \frac{1}{4+t^2}\,dt = \int_0^x \frac{1/4}{1 + (x/2)^2}\,dt = \frac{1}{2} \cdot \text{Arctan}\,\frac{x}{2}\Big]_0^x$$

$$= \frac{1}{2}\,\text{Arctan}\,\frac{x}{2}.$$

If we now evaluate $g(2)$ using both the series determined in part (c) and with the antiderivative $\frac{1}{2}\,\text{Arctan}\,\frac{x}{2}$, we obtain

$$\frac{2}{4} - \frac{8}{3 \cdot 16} + \frac{32}{5 \cdot 64} - \frac{128}{7 \cdot 256} + \ldots = \frac{1}{2}\,\text{Arctan}\,\frac{2}{2}$$

That is, $\frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \ldots = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$.